

ε -constants and equivariant Arakelov Euler characteristics

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1 Introduction

Let $R[G]$ be the group ring of a finite group G over a ring R . In this article, we study Euler characteristics of bounded metrised complexes of finitely generated $\mathbf{Z}[G]$ -modules, with applications to Arakelov theory and the determination of ϵ -constants.

A metric on a bounded complex K^\bullet of finitely generated $\mathbf{Z}[G]$ -modules is specified by giving for each irreducible character ϕ of G a metric on the determinant of the ϕ -isotypic piece of the complex $\mathbf{Q} \otimes_{\mathbf{Z}} K^\bullet$ of $\mathbf{Q}[G]$ -modules. Ignoring metrics for the moment, the alternating sum of the terms of K^\bullet yields an Euler characteristic in the Grothendieck group $G_0(\mathbf{Z}[G])$ of finitely generated $\mathbf{Z}[G]$ -modules. If K^\bullet is perfect, in the sense that all its terms are projective, one has an Euler characteristic in the finer Grothendieck group $K_0(\mathbf{Z}[G])$ of all finitely generated projective $\mathbf{Z}[G]$ -modules. To take metrics into account, we will use a metrized version $A(\mathbf{Z}[G])$ of the projective class group of $\mathbf{Z}[G]$. We will construct in $A(\mathbf{Z}[G])$ an “Arakelov-Euler characteristic” associated to each bounded perfect metrised complex of $\mathbf{Z}[G]$ -modules.

Our primary interest will be metrised complexes arising in the following way. Let \mathcal{X} be a scheme which is projective and flat over $\mathrm{Spec}(\mathbf{Z})$ and which has smooth generic fibre. We suppose that \mathcal{X} supports an action by a finite group G and that the action is tame in the sense that, for each closed point x of \mathcal{X} , the order of the inertia group of x is coprime to the residue characteristic of x . Choose a G -invariant Kähler metric h on the tangent bundle of the associated complex manifold $\mathcal{X}(\mathbf{C})$. We are then able to construct an Arakelov-Euler characteristic for any hermitian G -bundle (\mathcal{F}, j) on \mathcal{X} by endowing the equivariant determinant of cohomology of $R\Gamma(\mathcal{X}, \mathcal{F})$ with equivariant Quillen metrics $j_{Q, \phi}$ for each irreducible character ϕ of G . This construction can be extended to give an Arakelov-Euler

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characteristic for a bounded complex of hermitian G -bundles. Roughly speaking, our main results show that the Arakelov-Euler characteristic of the (logarithmic) de Rham complex of \mathcal{X} determines certain ε -constants which are associated to the L-functions of the Artin motives obtained from \mathcal{X} and the symplectic representations of G .

Let us now describe these results in more detail. Firstly we need to say a little more about arithmetic classgroups. Let R_G (resp. R_G^s) be the group of virtual characters (resp. virtual symplectic characters) of G . In Section 4 we obtain a quotient $A^s(\mathbf{Z}[G])$ of $A(\mathbf{Z}[G])$ called the symplectic arithmetic classgroup by restricting functions on R_G to R_G^s . We show that $A^s(\mathbf{Z}[G])$ contains a subgroup $R(\mathbf{Z}[G])$, called the group of rational classes, which supports a natural isomorphism $\theta : R(\mathbf{Z}[G]) \rightarrow \text{Hom}_{\text{Gal}}(R_G^s, \mathbf{Q}^\times)$. Now let S denote a finite set of primes which includes those primes where \mathcal{X} has non-smooth reduction. Let $\Omega_{\mathcal{X}/\mathbf{Z}}^1(\log \mathcal{X}_S^{\text{red}}/\log S)$ denote the sheaf of degree one relative logarithmic differentials of \mathcal{X} with respect to the morphism $(\mathcal{X}, \mathcal{X}_S^{\text{red}}) \rightarrow (\text{Spec}(\mathbf{Z}), S)$ of schemes with log-structures. This sheaf is locally free if (as we now assume) each special fibre of \mathcal{X} is a divisor with strictly normal crossings and the multiplicities of the irreducible components of each fibre are prime to the residue characteristic. The logarithmic de Rham complex $\Omega_{\mathcal{X}/\mathbf{Z}}^\bullet(\log \mathcal{X}_S^{\text{red}}/\log S)$ is defined to be the complex

$$\mathcal{O}_{\mathcal{X}} \rightarrow \Omega_{\mathcal{X}/\mathbf{Z}}^1(\log \mathcal{X}_S^{\text{red}}/\log S) \rightarrow \cdots \rightarrow \Omega_{\mathcal{X}/\mathbf{Z}}^d(\log \mathcal{X}_S^{\text{red}}/\log S)$$

We view each term of this complex as carrying the hermitian metric given by the corresponding exterior power of h^D . In Theorem 7.1 we completely describe the image \mathfrak{c}^s of the logarithmic de Rham Arakelov-Euler characteristic $\mathfrak{c} = \chi(\text{R}\Gamma(\mathcal{X}, \Omega_{\mathcal{X}/\mathbf{Z}}^\bullet(\log \mathcal{X}_S^{\text{red}}/\log S)), \wedge^\bullet h_Q^D)$ in the symplectic arithmetic classgroup $A^s(\mathbf{Z}[G])$. Here we limit ourselves to stating the result for characters of degree zero, since this result is particularly striking (see Theorem 7.1 for the result for characters of arbitrary degree):

Theorem. *The element \mathfrak{c}^s of $A^s(\mathbf{Z}[G])$ is a rational class and for any virtual symplectic character ψ of degree zero*

$$\theta(\mathfrak{c}^s)(\psi) = \varepsilon_0(\mathcal{Y}, \psi)^{-1}.$$

Whilst this result is of interest in itself, our principal concern rests with the sheaf of differentials $\Omega_{\mathcal{X}/\mathbf{Z}}^1$. One can associate to $\Omega_{\mathcal{X}/\mathbf{Z}}^1$ and the metric h^D a class Ω in the arithmetic Grothendieck group $\widehat{K}_0(\mathcal{X})$, which is a λ -ring by [GS1, §7]. In [CPT2] we show

$$\sum_{i=0}^d \chi(\lambda^i(\Omega)) = -\log|\varepsilon(\mathcal{X})|.$$

when G is trivial and \mathcal{X} satisfies the previous fibral hypotheses, where $\varepsilon(\mathcal{X})$ is the constant in the functional equation of the Hasse-Weil zeta function of \mathcal{X} . Furthermore, under much more general circumstances, we show that such an equality is equivalent to Bloch's Conjecture characterising the conductor of \mathcal{X}/\mathbf{Z} in terms of the top localised Chern class of the differentials of \mathcal{X}/\mathbf{Z} . The results of this present paper give equivariant refinements to the results of [CPT2]. Our main result, Theorem 8.3, shows that the equivariant de Rham Arakelov-Euler characteristic $\mathfrak{d} \in A(\mathbf{Z}[G])$ of \mathcal{X}/\mathbf{Z} , together with the arithmetic ramification class $\text{AR}(\mathcal{X}) \in A(\mathbf{Z}[G])$ (see 8.2), completely determines the ε -constants (together with their sign) for the symplectic representations of the group G , which acts on \mathcal{X} .

Theorem. *The element $\mathfrak{d}^s \cdot \text{AR}^s(\mathcal{X})^{-1}$ of $A^s(\mathbf{Z}[G])$ is a rational class and for any symplectic character ψ*

$$\theta(\mathfrak{d}^s \cdot \text{AR}^s(\mathcal{X})^{-1})(\psi) = \varepsilon(\mathcal{Y}, \psi)^{-1}.$$

This result can be thought of as a “converse” to the main Theorems of [CEPT1] and [CPT1]; there the class of the de Rham Euler characteristic in $K_0(\mathbf{Z}[G])$ is shown to be determined by ε -factors. Let us point out that the arithmetic ramification class $\text{AR}(\mathcal{X})$ is modeled on the “ramification class” $R(\mathcal{X}/\mathcal{Y})$ of [CEPT1]. It only depends on the branch locus of the cover $\mathcal{X} \rightarrow \mathcal{Y}$; under our assumptions this branch locus is contained on a finite set of fibers of $\mathcal{Y} \rightarrow \text{Spec}(\mathbf{Z})$.

In the classical case of the ring of integers \mathcal{O}_N of a tame Galois extension of number fields N/K with $G = \text{Gal}(N/K)$, which corresponds to $\mathcal{X} = \text{Spec}(\mathcal{O}_N)$, one has Fröhlich's hermitian conjecture (now the theorem of Cassou-Noguès-Taylor in [CNT]). Roughly speaking, this shows that the $\mathbf{Z}[G]$ -module \mathcal{O}_N together with the additional structure provided by the hermitian pairing of the trace form can be used to determine the symplectic ε -constants. A main observation of the present paper is that the role of the trace form is not so central; we can replace the trace form by Arakelov hermitian metrics. In fact, the arithmetic classgroup $A(\mathbf{Z}[G])$ may be considered as generalisation of Fröhlich's hermitian classgroup. There are, however, two crucial differences: firstly, we consider complex valued hermitian forms, whereas his forms are rational valued; the second key-feature of our approach is that it brings in all the characters of G , and not only the symplectic characters. From our point of view, the Cassou-Noguès-Taylor result may be reformulated as follows. They show that the signs of the Artin L-functions for symplectic representations of G can be recovered from the isomorphism class of \mathcal{O}_N as a metrised $\mathbf{Z}[G]$ -module. Here a metrised $\mathbf{Z}[G]$ -module is a $\mathbf{Z}[G]$ -module M together with a G -invariant metric on $\mathbf{C} \otimes_{\mathbf{Z}} M$. For the ring of integers

\mathcal{O}_N , the metric on $\mathbf{C} \otimes_{\mathbf{Z}} \mathcal{O}_N$ is given by $z \otimes a \mapsto (\sum_{\sigma} |z\sigma(a)|^2)^{1/2}$, where the sum extends over the distinct embeddings $\sigma : N \rightarrow \mathbf{C}$. The theorems in this article provide generalizations of this Cassou-Noguès-Taylor result to higher dimensions. Let us point out that, in higher dimensions, a partial result of similar flavour was obtained in [CEPT2]; however, the information, which was used there to characterise the ε_0 -constants, does not come directly from the de Rham cohomology.

This article is structured as follows: in Section 2 we define our notation and present a number of preliminary results. Then, in Section 3, we define the arithmetic classes for suitable bounded $\mathbf{Z}[G]$ -complexes and establish a number of their basic properties. The construction of the arithmetic class is a rather delicate matter, since we wish to produce an invariant which reflects the fact that the terms in the complex are projective, whilst the metrics are only defined on the determinants of cohomology. The main point here is to show that our notion of arithmetic class is invariant under quasi-isomorphisms which preserve metrics in an appropriate sense. The formation of arithmetic classes may also be seen to be closely related to the refined Euler characteristics with values in relative K-groups introduced by D. Burns in [Bu1] and [Bu2].

Our arithmetic classes take values in the arithmetic classgroup $A(\mathbf{Z}[G])$. In practice, it is often convenient to work with quotient groups of $A(\mathbf{Z}[G])$ such as those in Section 4.

In Section 5 we consider an arithmetic variety \mathcal{X} which carries a tame action by a finite group G and we define an arithmetic class for a hermitian G -bundle on \mathcal{X} which supports a set of metrics on the equivariant determinant of cohomology; we then carry out a number of calculations in the case when \mathcal{X} is the spectrum of a ring of integers. In Section 6 we fix a choice of Kähler metric on the tangent bundle of $\mathcal{X}(\mathbf{C})$. For a complex $(\mathcal{G}^{\bullet}, h_{\bullet})$ of hermitian G -bundles on \mathcal{X} , we use the equivariant Quillen metrics on the equivariant determinants of hypercohomology (see [B]) to construct an arithmetic class $\chi(\mathrm{R}\Gamma \mathcal{G}^{\bullet}, h_{Q\bullet})$; we call this class the Arakelov-Euler characteristic of $(\mathcal{G}^{\bullet}, h_{\bullet})$. We then briefly detail the functorial properties of such Euler characteristics and calculate such Euler characteristics when \mathcal{X} has dimension one. In Section 7 we describe the arithmetic class associated to the de Rham complex of logarithmic differentials; this then enables us to derive our characterisation of the symplectic ε_0 -constants of \mathcal{X} . Finally, in Section 8, we consider the de Rham Arakelov-Euler characteristic associated to the (regular) differentials of \mathcal{X}/\mathbf{Z} , and we show how this arithmetic class determines the symplectic ε -constants of \mathcal{X} .

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2 Preliminary results

2.A. Hermitian complexes.

Let R denote a commutative ring which is endowed with a fixed embedding into the field of complex numbers \mathbf{C} ; in applications R will be either \mathbf{Z} , \mathbf{R} or \mathbf{C} . We consider bounded (cochain) complexes K^\bullet of finitely generated left $R[G]$ -modules

$$K^\bullet : \dots \rightarrow K^i \xrightarrow{d_K^i} K^{i+1} \rightarrow \dots$$

so that the boundary maps d_K^i are all $R[G]$ -maps. Thus the i th cohomology group, denoted $H^i = H^i(K^\bullet)$, is an $R[G]$ -module. Recall that the complex K^\bullet is called *perfect* if in addition all the modules K^i are $R[G]$ -projective.

Definition 2.1 Let $x \mapsto \bar{x}$ denote the complex conjugation automorphism of \mathbf{C} ; we extend complex conjugation to an involution of the complex group algebra $\mathbf{C}[G]$ by the rule $\overline{\sum a_g g} = \sum \bar{a}_g g^{-1}$. An element $x \in \mathbf{C}[G]$ is called *symmetric* if $\bar{x} = x$. A *hermitian $R[G]$ -complex* is a pair (K^\bullet, k^\bullet) where K^\bullet is an $R[G]$ -complex, as above, and where each $K_{\mathbf{C}}^i = \mathbf{C} \otimes_R K^i$ is endowed with a non-degenerate positive-definite G -invariant hermitian form

$$k^i : K_{\mathbf{C}}^i \times K_{\mathbf{C}}^i \rightarrow \mathbf{C}.$$

Thus, in particular, each k^i is left \mathbf{C} -linear and $\overline{k^i(x, y)} = k^i(y, x)$. The metric associated to k^i is defined by $\|x\|^i = \sqrt{k^i(x, x)}$ for $x \in K_{\mathbf{C}}^i$, where $k^i(x, x) \geq 0$ because k^i is a positive definite hermitian form.

Equivalently (as per p 164 in [F2]) we may work with the $\mathbf{C}[G]$ -valued hermitian forms

$$\widehat{k}^i : K_{\mathbf{C}}^i \times K_{\mathbf{C}}^i \rightarrow \mathbf{C}[G]$$

given by the rule that for $x, y \in K_{\mathbf{C}}^i$

$$\widehat{k}^i(x, y) = \sum_{g \in G} k^i(x, gy)g.$$

Thus \widehat{k}^i is $\mathbf{C}[G]$ -left linear and is reflexive in the sense that $\overline{\widehat{k}^i(x, y)} = \widehat{k}^i(y, x)$. Conversely, given \widehat{k}^i , we may of course recoup k^i by reading off the coefficient of 1_G in $\mathbf{C}[G]$.

Example 2.2 The module $\mathbf{C}[G]$ carries the so-called *standard* positive G -invariant hermitian form

$$\mu : \mathbf{C}[G] \times \mathbf{C}[G] \rightarrow \mathbf{C}$$

given by the rule $\mu(\sum x_g g, \sum y_h h) = \sum x_g \bar{y}_g$. Then the associated $\mathbf{C}[G]$ -valued hermitian form $\hat{\mu}$

$$\hat{\mu} : \mathbf{C}[G] \times \mathbf{C}[G] \rightarrow \mathbf{C}[G]$$

is the so-called multiplication form $\hat{\mu}(x, y) = x \cdot \bar{y}$.

2.B. Metrised complexes.

Let G again denote a finite group, let \hat{G} denote the set of irreducible complex characters of G , and once and for all for each $\phi \in \hat{G}$ we let $W = W_\phi$ denote the simple 2-sided $\mathbf{C}[G]$ -ideal with character $\phi(1)\bar{\phi}$, where $\bar{\phi}$ is the contragredient character of ϕ . For a finitely generated $\mathbf{C}[G]$ -module M we define $M_\phi = (M \otimes_{\mathbf{C}} W)^G$, where G acts diagonally and on the left of each term; more generally, for a bounded complex P^\bullet of finitely generated $\mathbf{C}[G]$ -modules, we put $H^i = H^i(P^\bullet)$ and we write

$$P_\phi^\bullet = (P^\bullet \otimes_{\mathbf{C}} W)^G \quad \text{and} \quad H_\phi^i = (H^i \otimes_{\mathbf{C}} W)^G.$$

We then construct the complex lines $\det(P^\bullet(\phi))$ and $\det(H^\bullet(\phi))$ such that

$$\det(P^\bullet(\phi))^{\otimes \phi(1)} = \det(P_\phi^\bullet) = \otimes_i (\wedge^{\text{top}} P_\phi^i)^{(-1)^i}$$

$$\text{and} \quad \det(H^\bullet(\phi))^{\otimes \phi(1)} = \det(H_\phi^\bullet) = \otimes_i (\wedge^{\text{top}} H_\phi^i)^{(-1)^i}$$

where for a complex vector space V of dimension d , $\wedge^{\text{top}} V$ denotes $\wedge^d V$ and where for a complex line L we write L^{-1} for the dual line $\text{Hom}(L, \mathbf{C})$. Thus P_ϕ^\bullet should be thought of as corresponding to the character $\phi(1)\bar{\phi}$, whereas $P^\bullet(\phi)$ corresponds to the character $\bar{\phi}$. Observe that a metric on $\det(P_\phi^\bullet)$ determines a unique metric on $\det(P^\bullet(\phi))$ and of course conversely a metric on $\det(P^\bullet(\phi))$ determines a unique metric on $\det(P_\phi^\bullet)$. Note also that here and in the sequel for two finite dimensional vector spaces V_i of dimension d_i we normalise the standard isomorphism $\wedge^{d_1 d_2} (V_1 \otimes V_2) \cong \wedge^{d_1 d_2} (V_2 \otimes V_1)$ by multiplying by $(-1)^{d_1 d_2}$, in order to avoid subsequent sign complications. We refer to the set of lines $\det(H_\phi^\bullet)$ as the *equivariant determinants of cohomology* of P^\bullet . From Theorem 2 in [KM] we have a canonical isomorphism

$$\xi_\phi : \det(P_\phi^\bullet) \cong \det(H_\phi^\bullet). \tag{1}$$

For ease of computation we use the above definition of P_ϕ^\bullet ; however, alternatively one can also work with the isotypical components $\overline{W}P^\bullet$, as shown in the following lemma. Here

and in further applications we shall often need the renormalised form $\nu : \mathbf{C}[G] \times \mathbf{C}[G] \rightarrow \mathbf{C}$ of the hermitian form μ of (2.2) given by

$$\nu(x, y) = |G| \cdot \mu(x, y) \quad \text{for } x, y \in \mathbf{C}[G].$$

Lemma 2.3 *For a $\mathbf{C}[G]$ module V with a G -invariant metric $\|-\|$, the natural isomorphism*

$$\alpha : (V \otimes_{\mathbf{C}} W)^G \cong \overline{W}V$$

given by $\alpha(\sum_i v_i \otimes w_i) = \sum_i \overline{w_i} v_i$ is an isometry, where both terms carry the natural metrics induced by ν and $\|-\|$; that is to say $\overline{W}V$ carries the metric given by the restriction of $\|-\|$, and $(V \otimes_{\mathbf{C}} W)^G$ carries the metric given by the restriction of the tensor metric associated to $\|-\|$ and ν on $V \otimes \mathbf{C}[G]$.

PROOF. Let $\|-\|_1$ resp. $\|-\|_2$ denote the given metric on $(V \otimes_{\mathbf{C}} W)^G$ resp. $\overline{W}V$. If $e = |G|^{-1} \cdot \sum_g \phi(g) g$ is the central idempotent associated to W , then for $x \in \overline{W}V$, we have $\alpha^{-1}(x) = |G|^{-1} \cdot \sum_{g \in G} gx \otimes ge$ and so

$$\begin{aligned} \alpha^{-1}(x) &= \frac{1}{|G|^2} \sum_{g, h \in G} gx \otimes g\phi(h)h \\ &= \frac{1}{|G|^2} \sum_{g, h \in G} gh h^{-1} \phi(h) x \otimes gh \\ &= \frac{1}{|G|} \sum_{f \in G} f \bar{e} x \otimes f = \frac{1}{|G|} \sum_{f \in G} f x \otimes f. \end{aligned}$$

Thus

$$\|\alpha^{-1}(x)\|_1^2 = \frac{1}{|G|} \sum_g \|x\|_2^2 = \|x\|_2^2. \quad \square$$

Definition 2.4 Let R again denote a subring of \mathbf{C} . A *metrised $R[G]$ -complex* is a pair (P^\bullet, p_\bullet) , where P^\bullet is a bounded complex of finitely generated (not necessarily projective) $R[G]$ -modules and the p_ϕ are a set of metrics given by positive definite hermitian forms on the complex lines $\det(H_\phi^\bullet)$, one for each $\phi \in \widehat{G}$.

Let R_G denote the group of complex virtual characters of G . For each $\phi \in \widehat{G}$, let $p(\phi)$ denote the metric on $\det P^\bullet(\phi)$ induced by p_ϕ . For a virtual character $\chi = \sum_\phi n_\phi \phi \in R_G$, we write

$$\det(P^\bullet(\chi)) = \otimes_\phi \det(P^\bullet(\phi))^{\otimes n_\phi}$$

and we endow this complex line with the product metric $p(\chi) = \otimes_\phi p(\phi)^{n_\phi}$.

Example 2.5 A hermitian complex (K^\bullet, k^\bullet) affords a metrised complex in the following way: endow $(K^i \otimes_{\mathbf{C}} W)^G$ with the form induced by k^i on K^i and by the restriction of the standard form on W , which is given by the restriction of ν . The alternating tensor product of the top exterior products of these forms is then a positive definite hermitian form on the complex line $\det(K_\phi^\bullet)$ and so induces a positive definite hermitian form on the complex line $\det(H_\phi^\bullet)$ via (1); this then determines a unique metric on the complex line $\det(H^\bullet(\phi))$. Each such form determines a metric which we denote $\det(k(\phi))$.

3 Arithmetic Classes

3.A. The arithmetic classgroup.

In this sub-section we shall define the arithmetic classgroup in which our arithmetic classes take their values.

The notation is that of [CEPT2] and so we recall it only briefly: R_G denotes the group of complex characters of G ; $\overline{\mathbf{Q}}$ is the algebraic closure of \mathbf{Q} in \mathbf{C} , so that we have the inclusion map $\overline{\mathbf{Q}} \hookrightarrow \mathbf{C}$. We set $\Omega = \text{Gal}(\overline{\mathbf{Q}}/\mathbf{Q})$; J_f is the group of finite ideles in $\overline{\mathbf{Q}}$, that is to say the direct limit of the finite idele groups of all algebraic number fields E in $\overline{\mathbf{Q}}$.

Let $\widehat{\mathbf{Z}} = \prod_p \mathbf{Z}_p$ denote the ring of integral finite ideles of \mathbf{Z} . For $x \in \widehat{\mathbf{Z}}G^\times$, the element $\text{Det}(x) \in \text{Hom}_\Omega(R_G, J_f)$ is defined by the rule that for a representation T with character ψ

$$\text{Det}(x)(\psi) = \det(T(x));$$

the group of all such homomorphisms is denoted

$$\text{Det}(\widehat{\mathbf{Z}}G^\times) \subseteq \text{Hom}_\Omega(R_G, J_f).$$

More generally, for $n > 1$ we can form the group $\text{Det}(GL_n(\widehat{\mathbf{Z}}G))$; as each group ring $\mathbf{Z}_p[G]$ is semi-local we have the equality $\text{Det}(GL_n(\widehat{\mathbf{Z}}G)) = \text{Det}(\widehat{\mathbf{Z}}G^\times)$ (see 1.2.6 in [T2]).

For an $n \times n$ invertible matrix A with coefficients in $\mathbf{C}[G]$, $|\text{Det}(A)| \in \text{Hom}(R_G, \mathbf{R}_{>0})$ is defined by the rule

$$|\text{Det}(A)|(\psi) = |\text{Det}(A)(\psi)|.$$

Lemma 3.1 *Extending the involution $x \mapsto \bar{x}$ on $\mathbf{C}[G]$ to matrices over $\mathbf{C}[G]$ by transposition, for $\psi \in R_G$*

$$|\text{Det}(\overline{A})|(\psi) = |\text{Det}(A)|(\psi).$$

Replacing the ring $\widehat{\mathbf{Z}}$ by \mathbf{Q} , in the same way we construct

$$\mathrm{Det}(\mathbf{Q}[G]^\times) \subseteq \mathrm{Hom}_\Omega(R_G, \overline{\mathbf{Q}}^\times).$$

The product of the natural maps $\overline{\mathbf{Q}}^\times \rightarrow J_f$ and $|-| : \overline{\mathbf{Q}}^\times \rightarrow \mathbf{R}_{>0}$ yields an injection

$$\Delta : \mathrm{Det}(\mathbf{Q}[G]^\times) \rightarrow \mathrm{Hom}_\Omega(R_G, J_f) \times \mathrm{Hom}(R_G, \mathbf{R}_{>0}).$$

Definition 3.2 The *arithmetic classgroup* $A(\mathbf{Z}[G])$ is defined to be the quotient group

$$A(\mathbf{Z}[G]) = \left(\frac{\mathrm{Hom}_\Omega(R_G, J_f) \times \mathrm{Hom}(R_G, \mathbf{R}_{>0})}{\left(\mathrm{Det}(\widehat{\mathbf{Z}}[G]^\times) \times 1 \right) \mathrm{Im}(\Delta)} \right). \quad (2)$$

Remarks. (1) Note that, in the case when $G = \{1\}$, $A(\mathbf{Z})$ coincides with the usual Arakelov divisor class group of $\mathrm{Spec}(\mathbf{Z})$ (see 7.7 for further details).

(2) As indicated in the Introduction, there are two crucial differences between this arithmetic classgroup and the hermitian classgroup of Fröhlich (see II.5 in [F2]): firstly, we work with positive definite complex hermitian forms; secondly, as a consequence of this, we are able to work in a uniform manner with all characters of G .

3.B. The arithmetic class of a complex.

Let (P^\bullet, p_\bullet) be a perfect metrised $\mathbf{Z}[G]$ -complex; that is to say P^\bullet is a bounded metrised complex all of whose terms are finitely generated projective (and therefore locally free) $\mathbf{Z}[G]$ -modules. For each i , suppose that d_i is the rank of P^i as a $\mathbf{Z}[G]$ -module, and choose bases $\{a^{ij}\}$, resp. $\{\alpha_p^{ij}\}$ of

$$\mathbf{Q} \otimes P^i = \sum_j \mathbf{Q}[G] \cdot a^{ij}, \text{ resp. } \mathbf{Z}_p \otimes P^i = \sum_j \mathbf{Z}_p[G] \cdot \alpha_p^{ij}$$

over $\mathbf{Q}[G]$ resp. $\mathbf{Z}_p[G]$. As both $\{a^{ij}\}$ and $\{\alpha_p^{ij}\}$ are $\mathbf{Q}_p[G]$ -bases of $\mathbf{Q}_p \otimes P^i$, we can find $\lambda_p^i \in GL_{d_i}(\mathbf{Q}_p[G])$ such that $(a^{ij})_j = \lambda_p^i (\alpha_p^{ij})_j$, where $(a^{ij})_j$ denotes the column vector with j -th entry a^{ij} .

For $a \in \mathbf{Q} \otimes P^i$ we put

$$r(a) = \sum g a \otimes g \in P^i \otimes \mathbf{Q}[G]. \quad (3)$$

Note that for $h \in G$

$$r(ha) = r(a) (1 \otimes h^{-1}) \quad (4)$$

and for $w \in W$, the action of $r(a)$ on $1 \otimes w$ is defined to be

$$r(a)(1 \otimes w) = \sum_g ga \otimes gw \in (P^i \otimes W)^G. \quad (5)$$

For each $\phi \in \widehat{G}$ we choose an orthonormal basis $\{w_{\phi,k}\}$ of $W = W_\phi$ with respect to the standard form ν on $\mathbf{C}[G]$, then the $\{r(a^{ij})(1 \otimes w_{\phi,k})\}$ form a \mathbf{C} -basis of $(P^i \otimes W)^G$. By (4) and by linearity we have that for $\eta = \sum_{h \in G} \eta_h h \in \mathbf{Q}[G]$

$$r(\eta a)(1 \otimes w) = \sum_{h,g} \eta_h g h a \otimes gw = r(a)(1 \otimes \bar{\eta} w). \quad (6)$$

In the sequel for given i we shall write $\bigwedge r(a^i)(1 \otimes w_\phi)$ for the wedge product

$$\bigwedge_{j,k} (r(a^{ij})(1 \otimes w_{\phi,k})) \in \det(P^i \otimes W)^G.$$

We again adopt the notation of 2.B and let (P^\bullet, p_\bullet) be a perfect metrised $\mathbf{Z}[G]$ -complex; recall from (1) that for each $\phi \in \widehat{G}$ we have an isomorphism

$$\xi_\phi : \det(P_\phi^\bullet) \cong \det(H_\phi^\bullet).$$

Definition 3.3 With the above notation and hypotheses $\chi(P^\bullet, p_\bullet)$, the arithmetic class of (P^\bullet, p_\bullet) , is defined to be that class in $A(\mathbf{Z}[G])$ represented by the homomorphism on R_G which maps each $\phi \in \widehat{G}$ to the value in $J_f \times \mathbf{R}_{>0}$

$$\prod_p \left(\prod_i \text{Det}(\lambda_p^i)(\phi)^{(-1)^i} \right) \times p_\phi \left(\xi_\phi \left(\otimes_i \left(\bigwedge r(a^i)(1 \otimes w_\phi) \right)^{(-1)^i} \right) \right)^{\frac{1}{\phi(1)}}.$$

In the sequel we shall refer to the first coordinate as the *finite* coordinate and the second coordinate as the *archimedean* coordinate. In order to verify that this class is well-defined, we now show that it is independent of choices:

(i) If $\{\tilde{\alpha}_p^{ij}\}$ is a further set of $\mathbf{Z}_p[G]$ -bases for the $\mathbf{Z}_p \otimes P^i$, then we can find $z_p^i \in GL_{d_i}(\mathbf{Z}_p[G])$ such that

$$(\tilde{\alpha}_p^{ij})_j = z_p^i (\alpha_p^{ij})_j$$

and so the p -component of the finite coordinate of the homomorphism representing the class only changes by

$$\prod_i \text{Det}(z_p^i)^{(-1)^i} \in \text{Det}(\mathbf{Z}_p[G]^\times).$$

(ii) If $\{\tilde{a}^{ij}\}$ is a further set of $\mathbf{Q}[G]$ -bases for the $\mathbf{Q} \otimes P^i$, then we can find $\eta^i \in GL_{d_i}(\mathbf{Q}[G])$ such that

$$(\tilde{a}^{ij})_j = \eta^i(a^{ij})_j.$$

Now for each pair i, j , we have the equality $\tilde{a}^{ij} = \sum_l \eta_{jl}^i a^{il}$ and so by (6) we get

$$r(\tilde{a}^{ij})(1 \otimes w_{\phi,k}) = \sum_l r(a^{il})(1 \otimes \bar{\eta}_{jl}^i w_{\phi,k});$$

hence

$$\bigwedge r(\tilde{a}^i)(1 \otimes w_\phi) = \text{Det}(\bar{\eta}^i)(\bar{\phi})^{\phi(1)} \bigwedge r(a^i)(1 \otimes w_\phi).$$

As the η_{jl}^i have rational coefficients $\text{Det}(\bar{\eta}^i)(\bar{\phi}) = \text{Det}(\eta^i)(\phi)$, and so the homomorphism representing the class only changes by the homomorphism which maps ϕ to

$$\prod_i \text{Det}(\eta^i)(\phi)^{(-1)^i} \times \prod_i \left| \text{Det}(\eta^i)(\phi)^{(-1)^i} \right|;$$

and again this comes from an element of the denominator of (2).

(iii) If $\{\tilde{w}_{\phi,k}\}$ is a further orthonormal basis of W , then the wedge product $\bigwedge r(a^i)(1 \otimes \tilde{w}_\phi)$ differs from $\bigwedge r(a^i)(1 \otimes w_\phi)$ by a power of the determinant of a unitary base-change, which therefore has absolute value 1.

The following two properties of arithmetic classes follow readily from the definition.

Lemma 3.4 *$(P^\bullet, p_\bullet), (Q^\bullet, q_\bullet)$ be perfect metrisd $\mathbf{Z}[G]$ -complexes and endow the complex $P^\bullet \oplus Q^\bullet$ with metrics $p_\phi \otimes q_\phi$ on the equivariant determinants of cohomology via the identification*

$$\det(H^\bullet(P_\phi^\bullet \oplus Q_\phi^\bullet)) = \det(H^\bullet(P_\phi^\bullet)) \otimes \det(H^\bullet(Q_\phi^\bullet)).$$

Then

$$\chi(P^\bullet \oplus Q^\bullet, p_\phi \otimes q_\phi) = \chi(P^\bullet, p_\bullet) \chi(Q^\bullet, q_\bullet).$$

PROOF. This follows on choosing bases for P^\bullet and Q^\bullet and then using these bases to form a basis of $P^\bullet \oplus Q^\bullet$. \square

Recall that $|-|$ denotes the standard metric on \mathbf{C} .

Lemma 3.5 *If P^\bullet is an acyclic perfect metrisd $\mathbf{Z}[G]$ -complex and if we endow each complex line*

$$\det(H^\bullet(P_\phi^\bullet)) = \det(\{0\}) = \mathbf{C}$$

with the metric $|-|$, then $\chi(P^\bullet, |-|_\bullet) = 1$.

PROOF. As P^\bullet is acyclic and its terms are projective, it is isomorphic to a complex

$$\dots \rightarrow W^{i-1} \oplus W^i \rightarrow W^i \oplus W^{i+1} \rightarrow \dots$$

where the W^i are all projective and where the boundary maps are projection to the second factor. Using bases of the W^i to form bases of the P^i , together with the standard properties of \det , we see that the products in 3.3 all telescope to 1. \square

Lemma 3.6 *If p_\bullet and q_\bullet are two sets of metrics on the equivariant determinants of cohomology of P^\bullet , then for each $\phi \in \widehat{G}$, $p_\phi = \alpha(\phi)^{\phi(1)} q_\phi$ for a unique positive real number $\alpha(\phi)$. The class $\chi(P^\bullet, p_\bullet) \chi(P^\bullet, q_\bullet)^{-1}$ in $A(\mathbf{Z}[G])$ is represented by the homomorphism which maps each $\phi \in \widehat{G}$ to the value $1 \times \alpha(\phi)$.*

PROOF. This follows immediately from (3.3). \square

3.C Invariance under quasi-isomorphism.

Let (C^\bullet, c_\bullet) and (D^\bullet, d_\bullet) denote bounded (not necessarily perfect) metrised $\mathbf{Z}[G]$ -complexes and suppose that there is a $\mathbf{Z}[G]$ -cochain map $\alpha : C^\bullet \rightarrow D^\bullet$. Recall that α is called a *quasi-isomorphism* if it induces an isomorphism on the cohomology of the complexes. Theorem 2 in [KM] implies that if α is a quasi-isomorphism, then it induces natural isomorphisms

$$\det(H(\alpha_\phi)) : \det(H^\bullet(C_\phi^\bullet)) \cong \det(H^\bullet(D_\phi^\bullet))$$

so that the following square commutes:

$$\begin{array}{ccc} \det(C_\phi^\bullet) & \rightarrow & \det(D_\phi^\bullet) \\ \downarrow & & \downarrow \\ \det(H^\bullet(C_\phi^\bullet)) & \rightarrow & \det(H^\bullet(D_\phi^\bullet)) \end{array}$$

where the top horizontal map is $\det(\alpha_\phi)$ and where the vertical isomorphisms are $\xi_{C,\phi}$ and $\xi_{D,\phi}$ of (1).

Definition 3.7 A quasi-isomorphism $\alpha : C^\bullet \rightarrow D^\bullet$ is called a *metric quasi-isomorphism* from (C^\bullet, c_\bullet) to (D^\bullet, d_\bullet) if $c_\phi = d_\phi \circ \det(H(\alpha_\phi))$ for each $\phi \in \widehat{G}$.

The following result is an immediate consequence of the definitions:

Lemma 3.8 *Suppose again that $\alpha : C^\bullet \rightarrow D^\bullet$ is a quasi-isomorphic cochain map and that metrics d_ϕ are given on the $\det(H^\bullet(D_\phi^\bullet))$. Then there is a unique set of metrics c_ϕ on $\det(H^\bullet(C_\phi^\bullet))$ such that $\alpha : (C^\bullet, c_\bullet) \rightarrow (D^\bullet, d_\bullet)$ is a metric quasi-isomorphism; we call the*

metrics c_\bullet the metrics on the equivariant determinants of cohomology induced from d_\bullet via α . If $\beta : C^\bullet \rightarrow D^\bullet$ is a further quasi-isomorphic cochain map and if $H^i(\alpha) = H^i(\beta)$ for all i , then $\det(H(\alpha_\phi)) = \det(H(\beta_\phi))$ for all $\phi \in \widehat{G}$ and so α and β induce the same metrics on the equivariant determinant of cohomology of C^\bullet .

The main result of this sub-section is

Definition-Theorem 3.9 With the above notation and hypotheses, let $\alpha : (C^\bullet, c_\bullet) \rightarrow (D^\bullet, d_\bullet)$ be a metric quasi-isomorphism and suppose further that we can find perfect metrised $\mathbf{Z}[G]$ -complexes (P^\bullet, p_\bullet) , resp. (Q^\bullet, q_\bullet) which support metric quasi-isomorphisms $f : (P^\bullet, p_\bullet) \rightarrow (C^\bullet, c_\bullet)$, resp. $g : (Q^\bullet, q_\bullet) \rightarrow (D^\bullet, d_\bullet)$. Then $\chi(P^\bullet, p_\bullet) = \chi(Q^\bullet, q_\bullet)$.

In particular: for a metrised $\mathbf{Z}[G]$ -complex (C^\bullet, c_\bullet) with the property that C^\bullet is quasi-isomorphic to a perfect complex P^\bullet , we let p_\bullet denote the metrics on the equivariant determinant of cohomology of P^\bullet induced by c_\bullet ; then we can unambiguously define the arithmetic class of (C^\bullet, c_\bullet) to be the class $\chi(P^\bullet, p_\bullet)$; this class depends only on (C^\bullet, c_\bullet) and not on the particular choice of perfect complex P^\bullet . Thus with this definition we have the equality $\chi(C^\bullet, c_\bullet) = \chi(D^\bullet, d_\bullet)$.

Before proving the theorem we first need some preliminary results.

Lemma 3.10 Given maps of $\mathbf{Z}[G]$ -complexes $M^\bullet \xrightarrow{\varphi} L^\bullet \xleftarrow{\pi} N^\bullet$ with π a surjective quasi-isomorphism and with M^\bullet perfect, there exists a $\mathbf{Z}[G]$ -cochain map $\psi : M^\bullet \rightarrow N^\bullet$ such that $\pi \circ \psi = \varphi$.

PROOF. See VI.8.17 in [Mi]. \square

Corollary 3.11 If $0 \rightarrow A^\bullet \xrightarrow{\alpha} B^\bullet \xrightarrow{\beta} C^\bullet \rightarrow 0$ is an exact sequence of perfect $\mathbf{Z}[G]$ -complexes and if A^\bullet is acyclic, then there exists a cochain map $i : C^\bullet \rightarrow B^\bullet$ which is a section of β .

PROOF. Apply the above lemma to $C^\bullet = C^\bullet \xleftarrow{\beta} B^\bullet$. \square .

PROOF OF THEOREM. First we choose an acyclic perfect complex L^\bullet and a map $\lambda : L^\bullet \rightarrow D^\bullet$ such that $\lambda \oplus g$ is surjective. We then endow the equivariant determinants of the cohomology of L^\bullet with the trivial metrics l_\bullet as per Lemma 3.5. Then by 3.4 and 3.5

$$\chi(L^\bullet \oplus Q^\bullet, l_\bullet q_\bullet) = \chi(Q^\bullet, q_\bullet).$$

Thus, without loss of generality, we may now assume that g is surjective.

Consider the diagram

$$P^\bullet \xrightarrow{f} C^\bullet \xrightarrow{\alpha} D^\bullet \xleftarrow{g} Q^\bullet.$$

By Lemma 3.10 we can find a $\mathbf{Z}[G]$ -map $\beta : P^\bullet \rightarrow Q^\bullet$ such that $\alpha \circ f = g \circ \beta$. As f, g and α are all quasi-isomorphisms, β is also a quasi-isomorphism.

As previously, by adding an acyclic complex with trivial metrics (L^\bullet, l_\bullet) to (P^\bullet, p_\bullet) , setting $P'^\bullet = L^\bullet \oplus P^\bullet$ and $p'_\bullet = l_\bullet p_\bullet$, we obtain a surjective quasi-isomorphism $\beta' : P'^\bullet \rightarrow Q^\bullet$ and

$$\chi(P'^\bullet, p'_\bullet) = \chi(P'^\bullet, l_\bullet p_\bullet) = \chi(P^\bullet, p_\bullet).$$

We let $f' : P'^\bullet \rightarrow C^\bullet$ denote the composition of f with the natural projection map. Then in general of course it will not be true that $\alpha \circ f' = g \circ \beta'$; however, as L^\bullet is acyclic, we do know that $\alpha \circ f'$ and $g \circ \beta'$ agree on cohomology, i.e. $H^i(\alpha \circ f') = H^i(g \circ \beta')$ for all i .

In order to complete the proof of Theorem 3.9, we apply Corollary 3.11 to choose a section $\gamma : Q^\bullet \rightarrow P'^\bullet$ of β' . Again as per Lemma 3.5 we endow the equivariant determinants of cohomology of $\ker \beta'$ with the trivial metric s_\bullet ; as per Lemma 3.4 we endow P'^\bullet with the metric \tilde{q}_\bullet given by $s_\bullet \cdot \gamma_* q_\bullet$. Then

$$\chi(P'^\bullet, \tilde{q}_\bullet) = \chi(\ker \beta', s_\bullet) \chi(\gamma Q^\bullet, \gamma_* q_\bullet) = \chi(\gamma Q^\bullet, \gamma_* q_\bullet) = \chi(Q^\bullet, q_\bullet).$$

However, as the metrics q_\bullet , on the equivariant determinants of the cohomology of Q^\bullet , are induced from d_\bullet via $H(g)$, the metrics \tilde{q}_\bullet are the transport to P'^\bullet of the metrics d_\bullet via $H(g \circ \beta') = H(\alpha \circ f')$. Thus p'_\bullet and \tilde{q}_\bullet are both transports of the d_\bullet via $H(g \circ \beta') = H(\alpha \circ f')$, and so by Lemma 3.8 they are equal. Therefore we have shown

$$\chi(P^\bullet, p_\bullet) = \chi(P'^\bullet, p'_\bullet) = \chi(P'^\bullet, \tilde{q}_\bullet) = \chi(Q^\bullet, q_\bullet)$$

which is the desired result. \square

4 Arithmetic Classgroups

4.A Symplectic arithmetic classes.

The arithmetic classgroup $A(\mathbf{Z}[G])$ carries a great deal of information. In consequence, it is often advantageous in practice to work with various image groups. The most important of these is the *symplectic* arithmetic classgroup.

Recall that by the Hasse-Schilling norm theorem

$$\text{Det}(\mathbf{Q}[G]^\times) = \text{Hom}_\Omega^+(R_G, \overline{\mathbf{Q}}^\times) \quad (7)$$

where the right-hand expression denotes Galois equivariant homomorphisms whose values on R_G^s , the group of virtual symplectic characters, are all totally positive. By analogy with the map Δ of 3.A, we again have a diagonal map

$$\Delta^s : \text{Hom}_\Omega^+(R_G^s, \overline{\mathbf{Q}}^\times) \rightarrow \text{Hom}_\Omega(R_G^s, J_f) \times \text{Hom}(R_G^s, \mathbf{R}_{>0})$$

where $\Delta^s(f) = f \times |f| = f \times f$.

Definition 4.1 The group of *symplectic arithmetic classes* $A^s(\mathbf{Z}[G])$ is defined to be the quotient group

$$A^s(\mathbf{Z}[G]) = \frac{\text{Hom}_\Omega(R_G^s, J_f) \times \text{Hom}(R_G^s, \mathbf{R}_{>0})}{\text{Im} \Delta^s \cdot (\text{Det}^s(\widehat{\mathbf{Z}}[G]^\times) \times 1)}$$

where $\text{Det}^s(\widehat{\mathbf{Z}}[G]^\times)$ denotes the restriction of $\text{Det}(\widehat{\mathbf{Z}}[G]^\times)$ to R_G^s . In general, given a homomorphism f on R_G , we shall write f^s for the restriction of f to R_G^s . Clearly, restriction from R_G to R_G^s induces a homomorphism

$$\rho : A(\mathbf{Z}[G]) \rightarrow A^s(\mathbf{Z}[G]).$$

4.B Torsion classes.

Let $K_0\text{T}(\mathbf{Z}[G])$ denote the Grothendieck group of finite, cohomologically trivial $\mathbf{Z}[G]$ -modules and let $K_0\text{T}(\mathbf{Z}_p[G])$ denote the Grothendieck group of finite, cohomologically trivial $\mathbf{Z}_p[G]$ -modules. Thus the decomposition of a finite module into its p -primary parts induces the direct sum decomposition

$$K_0\text{T}(\mathbf{Z}[G]) = \oplus_p K_0\text{T}(\mathbf{Z}_p[G]).$$

We write $K_0(\mathbf{F}_p[G])$ for the Grothendieck group of finitely generated projective $\mathbf{F}_p[G]$ -modules; since each such module may be considered as a finite, cohomologically trivial $\mathbf{Z}_p[G]$ -module, we have a natural map

$$K_0(\mathbf{F}_p[G]) \rightarrow K_0\text{T}(\mathbf{Z}_p[G]).$$

From Chapter 1, Theorem 3.3 in [T2] recall that there is the Fröhlich isomorphism

$$K_0\text{T}(\mathbf{Z}[G]) \cong \frac{\text{Hom}_\Omega(R_G, J_f)}{\text{Det}(\widehat{\mathbf{Z}}[G]^\times)};$$

thus there is a natural map $\nu : K_0\text{T}(\mathbf{Z}[G]) \rightarrow A(\mathbf{Z}[G])$, induced by $f \mapsto f \times 1$ for $f \in \text{Hom}_\Omega(R_G, J_f)$.

In order that our invariants agree with the standard invariants in Arakelov theory, our convention here is that of I.3.2 in [T2]: namely, if $M = \mathbf{Z}_p[G]/\alpha\mathbf{Z}_p[G]$ is a \mathbf{Z}_p -torsion $\mathbf{Z}_p[G]$ -module, then the class of M in $K_0T(\mathbf{Z}[G])$ is represented by $\text{Det}(\alpha)$; this then is the inverse of the description given in 4.4 in [C]. It will be important in the sequel to keep this in mind when performing various torsion calculations in Sections 7 and 8.

4.C Tame arithmetic classes.

Although we shall ultimately always be interested in forming arithmetic classes over the integral group ring $\mathbf{Z}[G]$, in carrying out calculations it will often be advantageous to work with more general group rings, where we allow tame coefficients. With this in mind, we let T denote the maximal abelian tame extension of \mathbf{Q} in $\overline{\mathbf{Q}}$ and we set

$$\text{Det}^s(\widehat{\mathcal{O}}_T[G]^\times) = \varinjlim_L \text{Det}^s(\widehat{\mathcal{O}}_L[G]^\times)$$

where the direct limit extends over all finite extensions L of \mathbf{Q} in T and where $\widehat{\mathcal{O}}_L$ is the ring of integral adeles $\widehat{\mathbf{Z}} \otimes \mathcal{O}_L$.

Next we define the group $\text{Hom}_\Omega^+(R_G^s, \overline{\mathbf{Q}}^\times)$ to be the subgroup of $\text{Hom}_\Omega(R_G^s, \overline{\mathbf{Q}}^\times)$ whose values are all totally positive (note that the values of the homomorphisms in the latter group are of course all real by Galois equivariance); we then define Δ' to be the diagonal homomorphism

$$\text{Hom}_\Omega^+(R_G^s, \overline{\mathbf{Q}}^\times) \rightarrow \text{Hom}_\Omega(R_G^s, J_f) \times \text{Hom}(R_G^s, \mathbf{R}_{>0}).$$

In arithmetic calculations we shall often need to work with the *tame symplectic arithmetic classgroup* defined as

$$A_T^s(\mathbf{Z}[G]) = \frac{(\text{Det}^s(\widehat{\mathcal{O}}_T G^\times) \cdot \text{Hom}_\Omega(R_G^s, J_f)) \times \text{Hom}(R_G^s, \mathbf{R}_{>0})}{\text{Im}(\Delta') \cdot (\text{Det}^s(\widehat{\mathcal{O}}_T[G]^\times) \times 1)}. \quad (8)$$

Inclusion then induces a homomorphism

$$\eta : A^s(\mathbf{Z}[G]) \rightarrow A_T^s(\mathbf{Z}[G]).$$

For a perfect metrised $\mathbf{Z}[G]$ -complex (P^\bullet, p_\bullet) , we write $\chi^s(P^\bullet, p_\bullet)$ for the image of $\chi(P^\bullet, p_\bullet)$ in $A_T^s(\mathbf{Z}[G])$.

4.D Rational classes.

Rational classes are ubiquitous in arithmetic applications. The subgroup of *rational symplectic arithmetic classes* is defined to be the subgroup of $A_T^s(\mathbf{Z}[G])$ generated by

$\text{Hom}_\Omega(R_G^s, \mathbf{Q}^\times) \times 1$, that is to say

$$R^s(\mathbf{Z}[G]) = \frac{\left(\text{Hom}_\Omega(R_G^s, \mathbf{Q}^\times) \cdot \text{Det}^s(\widehat{\mathcal{O}_T[G]^\times}) \times 1 \right) \cdot \text{Im}(\Delta')}{\left(\text{Det}^s(\widehat{\mathcal{O}_T[G]^\times}) \times 1 \right) \cdot \text{Im}(\Delta')}.$$

The natural map $\overline{\mathbf{Q}}^\times \hookrightarrow J_f$ induces a map

$$\theta' : \text{Im}(\Delta') \cdot (\text{Hom}_\Omega(R_G^s, J_f) \times 1) \rightarrow \text{Hom}_\Omega(R_G^s, J_f)$$

which is defined as follows: consider $h \in \text{Im}(\Delta') \cdot (\text{Hom}_\Omega(R_G^s, J_f) \times 1)$ and let h_f resp. h_∞ denote the finite resp. archimedean component of h . Then h_∞ determines a unique element h'_∞ of $\text{Im}(\Delta')$; we define $\theta'(h) = h_f h'^{-1}_\infty$. Clearly θ' vanishes on $\text{Im}(\Delta')$ and so induces a homomorphism

$$\theta : R^s(\mathbf{Z}[G]) \rightarrow \frac{\text{Hom}_\Omega(R_G^s, \mathbf{Q}^\times) \text{Det}^s(\widehat{\mathcal{O}_T[G]^\times})}{\text{Det}^s(\widehat{\mathcal{O}_T[G]^\times})}.$$

From [CNT] (see also Corollary 3 to Theorem 17 in [F2]) we know that

$$\text{Hom}_\Omega(R_G^s, \mathbf{Q}^\times) \cap \text{Det}^s(\widehat{\mathcal{O}_T G^\times}) = \{1\} \quad (9)$$

and so by (9) we see that θ may be written as an isomorphism

$$\theta : R^s(\mathbf{Z}[G]) \rightarrow \text{Hom}_\Omega(R_G^s, \mathbf{Q}^\times).$$

4.E Passage to degree zero.

In this sub-section we describe a useful procedure for changing arithmetic classes by passage to characters of degree zero. In practice this will allow us to disregard various free classes which arise in our calculations. For an abelian group A and for $f \in \text{Hom}(R_G, A)$, we write $\tilde{f} \in \text{Hom}(R_G, A)$ for the homomorphism defined by the rule $\tilde{f}(\chi) = f(\chi - \chi(1)1_G)$, where 1_G denotes the trivial character of G . Note that for $z \in \mathbf{Z}_p[G]^\times$

$$\widetilde{\text{Det}}(z) = \text{Det}(zd^{-1}) \quad \text{where } d = \text{Det}(z)(1_G),$$

and so $\widetilde{\text{Det}(\mathbf{Z}_p[G]^\times)} \subset \text{Det}(\mathbf{Z}_p[G]^\times)$; similarly $\widetilde{\text{Im}(\Delta)} \subset \text{Im}(\Delta)$. Thus for a class $\mathfrak{c} \in A(\mathbf{Z}[G])$, represented by a homomorphism f under (2), we can unambiguously define a new class $\tilde{\mathfrak{c}}$, depending only on \mathfrak{c} , to be that class represented by the homomorphism \tilde{f} .

The map $\mathfrak{c} \mapsto \tilde{\mathfrak{c}}$ can be interpreted in the following fashion in terms of G -fixed points together with the induction map

$$\text{Ind} : A(\mathbf{Z}) \rightarrow A(\mathbf{Z}[G])$$

given in terms of character maps by $\text{Ind}(f)(\psi) = f(\text{Res}_G^{\{1\}}\psi) = f(\psi(1) \cdot 1_{\{1\}})$ for $\psi \in R_G$.

Lemma 4.2 *With the notation of 3.B, let $\mathfrak{c} = \chi(P^\bullet, p_\bullet) \in A(\mathbf{Z}[G])$ and let \mathfrak{c}_0 be the class in $A(\mathbf{Z})$ of $(P^{\bullet G}, p_1)$, where $P^{\bullet G}$ denotes the complex obtained from P^\bullet by taking G -fixed points and where p_1 denotes the metric on the determinant of the cohomology of $P^{\bullet G}$, obtained by identifying $P_{\mathbf{C}}^{\bullet G}$ with the isotypic component of the $P_{\mathbf{C}}^\bullet$ for the trivial character of G . There is then an equality $\tilde{\mathfrak{c}} = \mathfrak{c} \cdot \text{Ind}(\mathfrak{c}_0)^{-1}$ in $A(\mathbf{Z}[G])$.*

PROOF. Let $\Sigma = \sum_{g \in G} g$. As each term of P^\bullet is projective, $P^{\bullet G} = \Sigma P^\bullet$. We adopt the notation of 3.B and assume that f is the representative character map for the class $\mathfrak{c} = \chi(P^\bullet, p_\bullet)$ obtained by using local bases $\{\alpha_p^{ij}\}, \{a^{ij}\}$. Then we let h denote the representative for the class $\mathfrak{c}_0 = \chi(P^{\bullet G}, p_1)$ obtained by using local bases $\{\Sigma \alpha_p^{ij}\}, \{\Sigma a^{ij}\}$. To prove the lemma it will then suffice to show that $f(1_G) = h(1_{\{1\}})$.

We start by considering the non-archimedean coordinates. With the notation of 3.B we have $(a^{ij})_j = \lambda_p^i(\alpha_p^{ij})_j$, and so $(\Sigma a^{ij})_j = e \cdot \lambda_p^i(\Sigma \alpha_p^{ij})_j$ where $e = \Sigma / |G|$ is the idempotent associated to the trivial character of G . Since $\det(e \lambda_p^i) = \text{Det}(\lambda_p^i)(1_G)$, we conclude that the non-archimedean coordinates of f and h are equal.

To conclude we consider the archimedean coordinates. As e resp. 1 is a basis element of length 1 for the trivial isotypic component of $\mathbf{C}[G]$ resp. \mathbf{C} with respect to ν_G resp. ν_1 (see 2.3 for the definition of ν), then as in 3.B we see that the archimedean coordinate of $f(1_G)$ resp. $h(1_{\{1\}})$ is obtained by evaluating p_1 on the wedge product $\bigwedge \alpha_G(r_G(a^{ij} \otimes e))^{(-1)^i}$ resp. $\bigwedge \alpha_1(\Sigma a^{ij} \otimes 1)^{(-1)^i}$ (see 2.3 to recall the definition of α). Since

$$\alpha_G(r_G(a^{ij} \otimes e)) = \alpha_G(\Sigma a^{ij} \otimes e) = \Sigma a^{ij} = \alpha_1(\Sigma a^{ij} \otimes 1)$$

it follows that the archimedean coordinates of $f(1_G)$ and $h(1_{\{1\}})$ are also equal. \square

5 Arithmetic applications

5.A Preliminary results.

Let \mathcal{X} be a projective scheme over $\text{Spec}(\mathbf{Z})$ with structure morphism $f : \mathcal{X} \rightarrow \text{Spec}(\mathbf{Z})$. Suppose further that \mathcal{X} is flat over $\text{Spec}(\mathbf{Z})$ with equidimensional fibres of dimension d and that the generic fibre of \mathcal{X} is smooth. For the sake of brevity, in the sequel we shall refer to \mathcal{X} simply as an arithmetic variety. Suppose further that \mathcal{X} is endowed with an action (\mathcal{X}, G) by a given finite group G ; we shall suppose that the action is *tame*, in the sense that for each closed point x of \mathcal{X} the inertia group of x has order coprime to the residue

characteristic of x . Since \mathcal{X} is projective, the quotient scheme $\mathcal{Y} = \mathcal{X}/G$ is defined and we denote the quotient morphism by $\pi : \mathcal{X} \rightarrow \mathcal{Y}$. Let b denote the branch locus on \mathcal{Y} of the cover \mathcal{X}/\mathcal{Y} . From now on we shall suppose that \mathcal{Y} is connected and that the branch locus b is a Cartier divisor on \mathcal{Y} with strictly normal crossings. Since G acts tamely on \mathcal{X} , we note that by the valuative criterion for properness it follows that G must act freely on the generic fiber $\mathcal{X}_{\mathbf{Q}}$ (see 1.2.4(d) in [CEPT1]).

We now consider the construction of arithmetic classes for complexes of sheaves on \mathcal{X} . For a detailed account of the formation of Euler characteristics (without metrics) associated to a tame action, the reader is referred to [CEPT4]. Let \mathcal{F}^\bullet denote a bounded complex of coherent G - \mathcal{X} sheaves. Consider a G -stable open affine cover \mathcal{U} of \mathcal{X} and take the chain complex C^\bullet which is the associated simple complex to the double complex $C^\bullet(\mathcal{U}, \mathcal{F}^\bullet)$. There is an isomorphism on the derived category between C^\bullet and $\mathrm{R}\Gamma(\mathcal{X}, \mathcal{F}^\bullet)$ which induces isomorphisms

$$\det(\mathrm{H}^\bullet(\mathrm{R}\Gamma(\mathcal{X}, \mathcal{F}^\bullet))_\phi) \cong \det(\mathrm{H}^\bullet(C^\bullet)_\phi) \text{ for all } \phi \in \widehat{G}.$$

Lemma 5.1 *For C^\bullet as above, there is a perfect $\mathbf{Z}[G]$ -complex P^\bullet with a quasi-isomorphism $\gamma : P^\bullet \rightarrow C^\bullet$.*

PROOF. For full details we refer to the proof of Theorem 1.1 in [C]; so we shall now briefly only sketch the proof for the reader's convenience. From Lemma III.12.3 in [H] we may construct a quasi-isomorphism $\gamma_1 : P_1^\bullet \rightarrow C^\bullet$ where the complex P_1^\bullet is a bounded complex of finitely generated $\mathbf{Z}[G]$ -modules all of whose terms except the initial term, P_1^N say, are free $\mathbf{Z}[G]$ -modules. Since the mapping cylinder of $\alpha \circ \gamma_1$ is acyclic with all terms, except possibly P_1^N , being cohomologically trivial $\mathbf{Z}[G]$ -modules, we therefore deduce that P_1^N is a cohomologically trivial $\mathbf{Z}[G]$ -module, and it may therefore be written as the quotient of two projective $\mathbf{Z}[G]$ -modules; replacing P_1^N by this perfect complex of length 2 provides P^\bullet and γ . \square

Definition 5.2 Suppose now that we are given metrics h_ϕ on the $\det(\mathrm{H}^\bullet(\mathrm{R}\Gamma(\mathcal{X}, \mathcal{F}^\bullet))_\phi)$ for all $\phi \in \widehat{G}$. These metrics then induce metrics p_ϕ on $\det(\mathrm{H}^\bullet(P_\phi^\bullet))$ and by Theorem 3.9 we know that the arithmetic class $\chi(P^\bullet, p_\bullet)$ is independent of choices; we denote this class

$$\chi(\mathrm{R}\Gamma(\mathcal{X}, \mathcal{F}^\bullet), h_\bullet)$$

and the image of this class in the symplectic arithmetic classgroup $A^s(\mathbf{Z}[G])$ will be denoted $\chi^s(\mathrm{R}\Gamma(\mathcal{X}, \mathcal{F}^\bullet), h_\bullet)$.

The following results describe some basic properties of such arithmetic classes. The first two results follow immediately from 3.4 and 3.6.

Proposition 5.3 *Let $\mathcal{F}^\bullet, \mathcal{G}^\bullet$ be bounded complexes of coherent G - \mathcal{X} sheaves; let h_\bullet , resp. g_\bullet be metrics on the equivariant determinants of cohomology of $R\Gamma(\mathcal{X}, \mathcal{F}^\bullet)$, resp. $R\Gamma(\mathcal{X}, \mathcal{G}^\bullet)$. Then*

$$\chi(R\Gamma(\mathcal{X}, \mathcal{F}^\bullet \oplus \mathcal{G}^\bullet), h_\bullet g_\bullet) = \chi(R\Gamma(\mathcal{X}, \mathcal{F}^\bullet), h_\bullet) \cdot \chi(R\Gamma(\mathcal{X}, \mathcal{G}^\bullet), g_\bullet).$$

Proposition 5.4 *Let j_\bullet denote a further set of metrics on the equivariant determinants of cohomology of $R\Gamma(\mathcal{X}, \mathcal{F}^\bullet)$ and suppose that for each $\phi \in \widehat{G}$, $h_\phi = \alpha(\phi)^{\phi(1)} j_\phi$ for $\alpha(\phi) \in \mathbf{R}_{>0}$. Then the hermitian class*

$$\chi(R\Gamma(\mathcal{X}, \mathcal{F}^\bullet), h) \cdot \chi(R\Gamma(\mathcal{X}, \mathcal{F}^\bullet), j)^{-1}$$

is represented by the homomorphism which maps each $\phi \in \widehat{G}$ to

$$\varphi \longmapsto 1 \times \alpha(\phi).$$

Proposition 5.5 *If $0 \rightarrow \mathcal{F} \rightarrow \mathcal{G} \rightarrow \mathcal{H} \rightarrow 0$ is an exact sequence of coherent G - \mathcal{X} sheaves with metrics $f_\bullet, g_\bullet, h_\bullet$, on their equivariant determinants of cohomology, with the property that $f_\phi \otimes h_\phi = g_\phi$ under the isomorphisms*

$$\det(H^\bullet(\mathcal{F})_\phi) \otimes \det(H^\bullet(\mathcal{H})_\phi) \cong \det(H^\bullet(\mathcal{G})_\phi)$$

for each $\phi \in \widehat{G}$, then there is an equality of arithmetic classes

$$\chi(R\Gamma(\mathcal{X}, \mathcal{F}), f_\bullet) \cdot \chi(R\Gamma(\mathcal{X}, \mathcal{H}), h_\bullet) = \chi(R\Gamma(\mathcal{X}, \mathcal{G}), g_\bullet).$$

PROOF. Let \mathcal{U} denote a G -stable affine cover of \mathcal{X} . Then we get the associated exact sequence of Čech complexes

$$0 \rightarrow \mathcal{C}^\bullet(\mathcal{U}, \mathcal{F}) \rightarrow \mathcal{C}^\bullet(\mathcal{U}, \mathcal{G}) \rightarrow \mathcal{C}^\bullet(\mathcal{U}, \mathcal{H}) \rightarrow 0$$

For brevity we put $\mathcal{C}_1^\bullet = \mathcal{C}^\bullet(\mathcal{U}, \mathcal{F})$, $\mathcal{C}_2^\bullet = \mathcal{C}^\bullet(\mathcal{U}, \mathcal{G})$, $\mathcal{C}_3^\bullet = \mathcal{C}^\bullet(\mathcal{U}, \mathcal{H})$. As mentioned at the start of this section, since the G -action is tame, we can then find perfect $\mathbf{Z}[G]$ -complexes with surjective quasi-isomorphisms

$$P_2^{\bullet'} \xrightarrow{\gamma} C_2^\bullet, \quad P_3^\bullet \rightarrow C_3^\bullet.$$

We assert that we can construct a commutative diagram in which the vertical maps are all surjective quasi-isomorphisms and in which the P_i^\bullet are perfect $\mathbf{Z}[G]$ -complexes:

$$\begin{array}{ccccccc} 0 & \rightarrow & \mathcal{C}^\bullet(\mathcal{U}, \mathcal{F}) & \rightarrow & \mathcal{C}^\bullet(\mathcal{U}, \mathcal{G}) & \rightarrow & \mathcal{C}^\bullet(\mathcal{U}, \mathcal{H}) \rightarrow 0 \\ & & \uparrow & & \uparrow & & \uparrow \\ 0 & \rightarrow & P_1^\bullet & \rightarrow & P_2^\bullet & \rightarrow & P_3^\bullet \rightarrow 0 \end{array}$$

The result will then follow on taking bases for the P_j^i for $j = 1, 3$ and using these to form bases of the P_2^i .

We now briefly sketch the construction of P_1^\bullet and P_2^\bullet . By 3.10 we can find a cochain map β such that the following diagram commutes:

$$\begin{array}{ccc} C_2^\bullet & \rightarrow & C_3^\bullet \\ \uparrow \gamma & & \uparrow \\ P_2^{\bullet'} & \xrightarrow{\beta} & P_3^\bullet \end{array}$$

By adding a free acyclic complex to $P_2^{\bullet'}$ we may assume that β is surjective; this then implies that $\ker \beta$ is a perfect complex, and so the restriction of γ to $\ker \beta$ provides a quasi-isomorphism to C_1^\bullet . By adding a free acyclic complex to $\ker \beta$ we obtain a surjective quasi-isomorphism onto C_1^\bullet , and the resulting complex is denoted P_1^\bullet . \square

Proposition 5.6 *Suppose that the $G\mathcal{X}$ sheaf \mathcal{F} is fibral, that is to say it is supported over a finite set of primes S in $\mathrm{Spec}(\mathbf{Z})$. Then the equivariant determinants of cohomology all identify with the trivial complex line \mathbf{C} , which we endow with the standard metric $|-|$ and*

$$\chi(\mathrm{R}\Gamma(\mathcal{X}, \mathcal{F}), |-|_\bullet) = \nu \circ f_*^T(\mathcal{F})$$

where f_*^T denotes the composition of

$$\oplus_{p \in S} K_0(G, \mathcal{X}_p) \xrightarrow{\oplus f_{p*}} \oplus_{p \in S} K_0(\mathbf{F}_p[G]) \rightarrow K_0\mathrm{T}(\mathbf{Z}[G]).$$

Here the first map is induced by the structure maps $f_p : \mathcal{X}_p \rightarrow \mathrm{Spec}(\mathbf{F}_p)$ for $p \in S$ (see Theorem 1.1 in [C]), and the second map is as described in 4.B.

PROOF. This follows at once from the definition of $\chi(\mathrm{R}\Gamma(\mathcal{X}, \mathcal{F}), |-|_\bullet)$ and from 4.B. (Note that this, in part, justifies the choice of convention in 4.B.) \square

5.B Rings of integers.

The remainder of this article is devoted to the study of images of arithmetic classes in various arithmetic situations. In this sub-section we shall consider the case where \mathcal{X} is the

spectrum of a ring of integers; thus in this sub-section we consider the case $\mathcal{X} = \text{Spec}(\mathcal{O}_N)$ for a ring of integers \mathcal{O}_N of a number field N which is at most tamely ramified over a number field K , with N/K Galois and $G = \text{Gal}(N/K)$.

Our main result here is Theorem 5.9, which is closely related to the work of Fröhlich in Chap. VI of [F2] and to the proof of the Second Fröhlich Conjecture in [CNT].

Suppose that \mathfrak{a} is a G -stable \mathcal{O}_N -ideal and let $\mathcal{F} = \tilde{\mathfrak{a}}$ be the associated G - \mathcal{X} sheaf viewed as a complex concentrated in degree zero. As \mathcal{X} is affine

$$H^i(\mathcal{X}, \mathcal{F}) = \begin{cases} \mathfrak{a} & \text{if } i = 0 \\ \{0\} & \text{if } i > 0. \end{cases}$$

We endow $\mathfrak{a}_{\mathbf{C}} = \mathbf{C} \otimes_{\mathbf{Z}} \mathfrak{a} = \mathbf{C} \otimes_{\mathbf{Q}} N$ with the G -invariant positive definite *Hecke form* $h : \mathbf{C} \otimes_{\mathbf{Q}} N \times \mathbf{C} \otimes_{\mathbf{Q}} N \rightarrow \mathbf{C}$ which is defined by the rule

$$h(\lambda \otimes m, \nu \otimes n) = \lambda \bar{\nu} \sum_{\sigma} \sigma(m) \overline{\sigma(n)}$$

where the sum extends over the embeddings $\sigma : N \rightarrow \mathbf{C}$. Thus, as in (2.5), h determines metrics on the $\det((\mathbf{C} \otimes_{\mathbf{Q}} N)_{\phi})$ for $\phi \in \widehat{G}$; we denote this set of metrics by $\det h_{\bullet}$.

Remark: We refer to the form h as the Hecke form since this form was introduced by Hecke in his proof of the functional equation for L-functions; see for instance 9.3 in [He].

We write μ_K for the G -invariant positive hermitian form on $\mathbf{C} \otimes_{\mathbf{Q}} K[G]$ given by the rule

$$\mu_K(\sum_g x_g g, \sum_h y_h h) = \sum_{\rho} \sum_{g,h} \delta_{g,h\rho} (x_g) \overline{\rho(y_h)}$$

where the first right-hand sum extends over all embeddings ρ of K into \mathbf{C} . Again as per (2.5) μ_K induces metrics $\det \mu_{K,\phi}$ on the $\det((\mathbf{C} \otimes_{\mathbf{Q}} K[G])_{\phi})$ for each $\phi \in \widehat{G}$; we denote this set of metrics by $\det \mu_{K,\bullet}$, or $\det \mu_{\bullet}$ when K is clear from the context.

In the sequel, since $\mathcal{X} = \text{Spec}(\mathcal{O}_N)$ is affine, for brevity we shall write $\chi(\mathfrak{a}, \det h_{\bullet})$ in place of $\chi(R\Gamma(\mathcal{X}, \mathfrak{a}), \det h_{\bullet})$ etc. The following result is an equivariant version of the usual discriminant-index theorem:

Proposition 5.7 *The following equality holds in $A(\mathbf{Z}[G])$*

$$\chi(\mathcal{O}_N, \det h_{\bullet}) \cdot \chi(\mathfrak{a}, \det h_{\bullet})^{-1} = \nu(\mathcal{O}_N/\mathfrak{a})$$

where ν is the map on torsion classes of 4.B.

PROOF. This follows from Propositions 5.5 and 5.6 applied to the exact sequence

$$0 \rightarrow \mathfrak{a} \rightarrow \mathcal{O}_N \rightarrow \mathcal{O}_N/\mathfrak{a} \rightarrow 0. \quad \square$$

Definition 5.8 For a given prime ideal \mathfrak{p} of \mathcal{O}_K , let $f_{\mathfrak{p}}$ denote the residue class extension degree of \mathfrak{p} in K/\mathbf{Q} , denote by $I_{\mathfrak{p}}$ the inertia group of a chosen prime ideal of \mathcal{O}_N above \mathfrak{p} ; and let $u_{\mathfrak{p}}$ denote the augmentation character of $I_{\mathfrak{p}}$ (that is to say the regular character minus the trivial character). Define

$$\mathrm{Pf}_p(\mathcal{O}_N) : R_G^s \rightarrow (-p)^{\mathbf{Z}} \text{ by the rule } \mathrm{Pf}_p(\mathcal{O}_N)(\psi) = \prod_{\mathfrak{p}|p} (-p)^{\frac{1}{2}f_{\mathfrak{p}}(\psi, \mathrm{Ind}_{I_{\mathfrak{p}}}^G u_{\mathfrak{p}})}$$

for $\psi \in R_G^s$, where $(,)$ denotes the standard inner product on R_G . Note that $(\psi, \mathrm{Ind}_{I_{\mathfrak{p}}}^G u_{\mathfrak{p}}) = (\psi|_{I_{\mathfrak{p}}}, u_{\mathfrak{p}})$ is an even integer, since $\psi|_{I_{\mathfrak{p}}}$ is a symplectic character of the cyclic group $I_{\mathfrak{p}}$ and is therefore a sum of characters of the form $\theta + \bar{\theta}$. We then define $\mathrm{Pf}(\mathcal{O}_N)$ to be the idele valued function, defined on symplectic characters, which is $\mathrm{Pf}_p(\mathcal{O}_N)$ at primes over p and which is 1 at the archimedean primes.

Let $\delta_K \in \mathrm{Hom}(R_G^s, \mathbf{R}_{>0})$ be the homomorphism

$$\delta_K(\psi) = (|G|^{[K:\mathbf{Q}]} |d_K|)^{\psi(1)/2}$$

where d_K is the discriminant of K/\mathbf{Q} .

The main result of this sub-section is the following description of the tame arithmetic classes $\chi^s(\mathcal{O}_N, \det h_{\bullet})$ and $\chi^s(\mathcal{O}_K G, \det \mu_{K\bullet})$.

Theorem 5.9 (a) The class $\chi^s(\mathcal{O}_N, \det h_{\bullet})$ in $A_T^s(\mathbf{Z}[G])$ is represented by the homomorphism $\tilde{\varepsilon}_{\infty}^s(K)^{-1} \mathrm{Pf}(\mathcal{O}_N)^{-1} \times \delta_K$, where, for a symplectic character ψ , $\tilde{\varepsilon}_{\infty}^s(K)(\psi)$ is the archimedean epsilon factor $\varepsilon_{\infty}(K, \psi - \psi(1) \cdot 1)$ (see Section 7);

(b) the class $\chi^s(\mathcal{O}_K G, \det \mu_{K\bullet})$ in $A_T^s(\mathbf{Z}[G])$ is represented by the homomorphism $1 \times \delta_K$.

Before proceeding with the proof of the theorem, we first introduce some notation and establish some preparatory results.

For a prime number p , let β_p be an $\mathcal{O}_{K,p}[G]$ -basis of $\mathcal{O}_{N,p}$ and let b be a $K[G]$ -basis of N (so that b is a so-called normal basis of N/K). Recall (see I.4 of [F2]) that for a character ψ of G the resolvent $(b | \psi)$ is defined to be the value $\mathrm{Det}(\sum_{g \in G} g(b)g^{-1})(\psi)$; note that, with the notation of (3) in 3.B, $(b | \psi) = \mathrm{Det}(r(b))(\bar{\psi})$ and so by (4) we have proved the following particular instance of the Galois action formula for resolvents (cf. Theorem 20A in [F1])

$$(g(b) | \psi) = (b | \psi) \cdot \det(\psi)(g). \quad (10)$$

The local resolvents $(\alpha_p | \psi)$ are defined similarly (see *loc. cit.*).

Set $\Omega_K = \text{Gal}(\overline{\mathbf{Q}}/K)$ and recall that we write Ω for $\Omega_{\mathbf{Q}}$. For an Ω -module A , let

$$\mathcal{N}_{K/\mathbf{Q}} : \text{Hom}_{\Omega_K}(R_G, A) \rightarrow \text{Hom}_{\Omega}(R_G, A)$$

denote the co-restriction map of (3.3) in II.3 of [F1]; we extend the domain of this map to include resolvents, which are not in general Ω_K -equivariant, as per (3.1) in III.3 in [F1].

Next we recall the p -adic absolute value function from [CEPT2]. Let $L = \mathbf{Q}(\zeta_p)$. By Lemma 3.1 *loc. cit* we know that we can find $\lambda \in L_p$ such that $\lambda^{p-1} = -p$. Once and for all we fix such a choice of λ and define $\|-\| : \overline{\mathbf{Q}}_p^{\times} \rightarrow \lambda^{\mathbf{Q}}$ by stipulating that $x \cdot \|x\|$ is to be a p -adic unit.

We now recall some related constructions from [CEPT2]; for full details see (3.1) and (3.2) in *loc. cit*. Let $R_G(\overline{\mathbf{Q}}_p)$ denote the ring of $\overline{\mathbf{Q}}_p$ -characters of G and set $\Omega_p = \text{Gal}(\overline{\mathbf{Q}}_p/\mathbf{Q}_p)$. For $g \in \text{Hom}(R_G(\overline{\mathbf{Q}}_p), \overline{\mathbf{Q}}_p^{\times})$ define $\|g\|(\varphi) = \|g(\varphi)\|$. We shall say that g is *well-defined* if $\|g\|$ takes values in $\lambda^{\mathbf{Z}}$.

Once and for all we fix a field embedding $h : \overline{\mathbf{Q}} \rightarrow \overline{\mathbf{Q}}_p$. From II.2.1 in [F1], h induces an isomorphism

$$h^* : \text{Hom}_{\Omega_L}(R_G, (\overline{\mathbf{Q}} \otimes \mathbf{Q}_p)^{\times}) \cong \text{Hom}_{\Omega_{L_p}}(R_G(\overline{\mathbf{Q}}_p), \overline{\mathbf{Q}}_p^{\times}).$$

For $f \in \text{Hom}_{\Omega_L}(R_G, (\overline{\mathbf{Q}} \otimes \mathbf{Q}_p)^{\times})$ define $\|f\| = h^{*-1}(\|h^*f\|)$; we shall say that $\|f\|$ is well-defined when $\|h^*f\|$ is well-defined.

In the sequel we employ a standard abuse of notation and write $\text{Det}(\mathcal{O}_{L_p}[G]^{\times})$ for $h^*(\text{Det}(\mathcal{O}_{L_p}[G]^{\times}))$.

Theorem 5.10 *For $\psi \in R_G^s$, $\text{sign}(\mathcal{N}_{K/\mathbf{Q}}(b \mid \psi)) = \varepsilon_{\infty}(K, \psi - \psi(1)1)$.*

PROOF. This is III.4.9 of [F1]. \square

Proposition 5.11 *We have $\mathcal{N}_{K/\mathbf{Q}}(\beta_p \mid -)^s \cdot \text{Pf}_p(\mathcal{O}_N)^{-1} \in \text{Det}^s(\mathcal{O}_{T,p}G^{\times})$ where we recall from 4.B that T denotes the maximal abelian tame extension of \mathbf{Q} in $\overline{\mathbf{Q}}$.*

PROOF. Let τ^* denote the adjusted Galois Gauss sum of (3.9) in [T1] (or see IV.1.7. in [F1]). From the discussion following Theorem 2 in [T1] we know that we can find $z_p \in \mathbf{Z}[G]^{\times}$ such that for all $\phi \in R_G$

$$\mathcal{N}_{K/\mathbf{Q}}(\beta_p \mid \phi) = \text{Det}(z_p)(\phi)\tau^*(\phi).$$

Recall that we have fixed a choice of field embedding $h : \overline{\mathbf{Q}} \rightarrow \overline{\mathbf{Q}}_p$. By Theorem 4 in [CEPT2] we know that $\|\tau_p^*\| = \|\tilde{\varepsilon}_{0,p}\|$ is well-defined. Writing

$$\tau^* = \tau' \tau_p^* \text{ where } \tau' = \prod_{q \neq p} \tau_q^*$$

we get

$$\mathcal{N}_{K/\mathbf{Q}}(\beta_p \mid -) = \text{Det}(z_p) \tau' \tau_p^* = \text{Det}(z_p) \tau' \tau_p^* \|\tau_p^*\| \|\tau_p^*\|^{-1}$$

and by Theorem 4 in *loc. cit*

$$\tau_p^* \|\tau_p^*\| \in \text{Det}(\mathcal{O}_{L_p}[G]^\times) \quad \text{and} \quad \tau' \in \text{Det}(\mathcal{O}_{T,p}[G]^\times).$$

From Theorem 7.4 in [M] we know that each value of τ_p^s is plus or minus an integral power of p . Thus for a symplectic character ψ of G ,

$$|\tau_p^*(\psi)| = N_{\mathfrak{f}_p}(\psi)^{\frac{1}{2}} = \prod_{\mathfrak{p}|p} N_{\mathfrak{p}}^{\frac{1}{2}(\psi, \text{Ind}_{\mathfrak{p}}^G \mathbf{u}_{\mathfrak{p}})} = \pm \text{Pf}_p(\mathcal{O}_N)(\psi).$$

where $N_{\mathfrak{f}_p}(\psi)$ denotes the p -part of the absolute norm of the Artin conductor of ψ . As $\|\tau_p^{*s}\|$ and $\text{Pf}_p(\mathcal{O}_N)$ are both integral powers of $-p$, we deduce that $\|\tau_p^{*s}\|^{-1} = \text{Pf}_p(\mathcal{O}_N)$ as required. \square

PROOF OF THEOREM 5.9. We begin by proving (a). Let $\{x_i\}$ denote a \mathbf{Z} -basis of \mathcal{O}_K . Then $\{x_i b\}$ resp. $\{x_i \beta_p\}$ is a $\mathbf{Q}[G]$ -basis resp. a $\mathbf{Z}_p[G]$ -basis for N resp. $\mathcal{O}_{N,p}$. With the previous notation choose $\lambda_p \in K_p[G]$ such that $b = \lambda_p \beta_p$ and write $x_i \lambda_p = \sum_j \lambda_p^{ij} x_j$ with $\lambda_p^{ij} \in \mathbf{Q}_p[G]$. Then

$$x_i b = x_i \lambda_p \beta_p = \sum_j \lambda_p^{ij} x_j \beta_p$$

and so the matrix $(\lambda_p^{ij})_{ij}$ transforms the $\mathbf{Q}_p[G]$ -basis $\{x_i \beta_p\}$ into the basis $\{x_i b\}$; therefore the finite coordinate of the representing homomorphism of the arithmetic class $\chi(\mathcal{O}_N, \det h_\bullet)$ is

$$\prod_p \text{Det}(\lambda_p^{ij}) = \prod_p \mathcal{N}_{K/\mathbf{Q}}(\lambda_p) = \prod_p \mathcal{N}_{K/\mathbf{Q}}(b \mid -) \cdot \mathcal{N}_{K/\mathbf{Q}}(\beta_p \mid -)^{-1}.$$

To obtain the archimedean coordinate for a chosen irreducible character ϕ we have to extend our notation and choose a positive integer n_ϕ such that $\det(\phi)^{n_\phi \phi(1)}$ is trivial. We then write $\psi = n_\phi \phi(1) \bar{\phi}$ and set

$$W_\psi = W_\phi^{n_\phi}$$

where $W_\phi^{n_\phi}$ denotes the direct sum of n_ϕ copies of W_ϕ . We endow W_ψ with the hermitian form, ν_ψ say, given by the orthogonal sum of the hermitian forms on the W_ϕ , and we let $\{w_{\psi,k}\}$ denote the basis of W_ψ derived from the bases $\{w_{\phi,l}\}$ of W_ϕ . We must now consider the wedge product

$$\bigwedge_{i,k} (x_i \cdot r(b)(1 \otimes w_{\psi,k})) = \bigwedge_i \bigwedge_k \left(\sum_g x_i g(b) \otimes g w_{\psi,k} \right) = \bigwedge_{i,k} y_i (1 \otimes w_{\psi,k}) \quad (11)$$

where $y_i = x_i(b \mid \psi)$; here we obtain the second equality from the fact that

$$\bigwedge_k \left(\sum_g g(b) \otimes gw_{\psi k} \right) = \text{Det}(r(b))(\overline{\psi}) \bigwedge_k w_{\psi k} = (b \mid \psi) \bigwedge_k w_{\psi k}.$$

A priori $(b \mid \psi) \in \mathbf{C} \otimes_{\mathbf{Q}} N$; however, because $\det(\psi) = 1$, by the Galois action formula (10), $(b \mid \psi) \in \mathbf{C} \otimes_{\mathbf{Q}} K$. Therefore

$$\bigwedge_i x_i(b \mid \psi) = N_{K/\mathbf{Q}}(b \mid \psi) \bigwedge_i x_i.$$

To complete the proof of (a), note first that as the $\{x_i\}$ are fixed by G and as $\{w_{\psi,k}\}$ is an orthonormal basis for the form ν_{ψ} ,

$$h \otimes \nu_{\psi}(x_i \otimes w_{\psi,k}, x_j \otimes w_{\psi,l}) = \sum_{\sigma} \sigma(x_i) \overline{\sigma(x_j)} \delta_{k,l} = |G| \sum_{\rho} \rho(x_i) \overline{\rho(x_j)} \delta_{k,l}. \quad (12)$$

In the sequel we shall write $(h \otimes \nu_{\psi})^G$ for the restriction of $h \otimes \nu_{\psi}$ from $(\mathbf{C} \otimes_{\mathbf{Q}} N) \otimes W$ to $((\mathbf{C} \otimes_{\mathbf{Q}} N) \otimes W)^G$. Hence the archimedean coordinate of the representing homomorphism of $\chi(\mathcal{O}_N, \det h_{\bullet})$ at ϕ is the $n_{\phi}\phi(1)$ -st root of

$$\begin{aligned} & \det(h_{\psi}(\bigwedge_{i,k} x_i r(b)(1 \otimes w_{\psi,k}))) = \\ & = \det \left((h \otimes \nu_{\psi})^G(x_i r(b) \otimes w_{\psi,k}, x_j r(b) \otimes w_{\psi,l}) \right)^{1/2} \\ & = |\mathcal{N}_{K/\mathbf{Q}}(b \mid \psi)| \det \left((h \otimes \nu_{\psi})^G(x_i \otimes w_{\psi,k}, x_j \otimes w_{\psi,l}) \right)^{1/2} \\ & = \left(|G|^{[K:\mathbf{Q}]} |d_K| \right)^{\psi(1)/2} |\mathcal{N}_{K/\mathbf{Q}}(b \mid \psi)|. \end{aligned}$$

Note that the square roots in the above right-hand terms (which are of course taken to be positive) arise since we are dealing with the metrics which are, of course, given by the square root of the corresponding positive definite hermitian forms. This then shows that the class $\chi(\mathcal{O}_N, \det h_{\bullet})$ is represented by the homomorphism which maps an irreducible character ϕ to the value

$$\mathcal{N}_{K/\mathbf{Q}}(b \mid \phi) \cdot \mathcal{N}_{K/\mathbf{Q}}(\beta_p \mid \phi)^{-1} \times |\mathcal{N}_{K/\mathbf{Q}}(b \mid \phi)| \cdot \delta_K(\phi).$$

We now consider $\chi^s(\mathcal{O}_N, \det h_{\bullet})$. Then by Proposition 5.10 this class is represented by the homomorphism which maps a symplectic character ψ to the value

$$\mathcal{N}_{K/\mathbf{Q}}(b \mid \psi) \mathcal{N}_{K/\mathbf{Q}}(\beta_p \mid \psi)^{-1} \times \tilde{\varepsilon}_{\infty}(K, \psi) \mathcal{N}_{K/\mathbf{Q}}(b \mid \psi) \delta_K(\psi).$$

Since

$$(\psi \mapsto \tilde{\varepsilon}_\infty(K, \psi) \cdot \mathcal{N}_{K/\mathbf{Q}}(b \mid \psi)) \in \text{Hom}_\Omega^+(R_G^s, \overline{\mathbf{Q}}^\times) = \text{Det}^s(\mathbf{Q}[G]^\times),$$

we conclude that the class is also represented by

$$\psi \mapsto \tilde{\varepsilon}_\infty(K, \psi)^{-1} \prod_p \mathcal{N}_{K/\mathbf{Q}}(\beta_p \mid \psi)^{-1} \times \delta_K(\psi)$$

and the result then follows from 5.11.

The proof for (b) is similar, but considerably easier, because we may replace b and all the β_p by 1 throughout in the above. Indeed, we see immediately that, with these choices, the finite coordinate is 1. Since

$$\begin{aligned} & (\mu_K \otimes \nu_\phi)^G \left(\sum_g x_i g \otimes gw_{\phi,k}, \sum_h x_j h \otimes hw_{\phi,l} \right) \\ &= \sum_\rho \sum_{g,h} \rho(x_i) \overline{\rho(x_j)} \mu(g, h) \nu(gw_{\phi,k}, hw_{\phi,l}) \\ &= \delta_{k,l} |G| \sum_\rho \rho(x_i) \overline{\rho(x_j)} \end{aligned}$$

we have

$$\det \mu_{K,\phi} \left(\bigwedge_{i,k} \sum_g x_i g \otimes gw_{\phi,k} \right) = \delta_K(\phi). \quad \square$$

6 Equivariant Quillen Metrics

6.A Definition of arithmetic classes.

In this section we consider an arithmetic variety \mathcal{X} with fibral dimension d . Since G acts tamely on \mathcal{X} , G must act freely on the complexified generic fibre $X := \mathcal{X} \times_{\mathbf{Z}} \mathbf{C}$; in the sequel we shall abuse terminology and identify X with the complex manifold $\mathcal{X}(\mathbf{C})$ of its complex points. We fix a G -invariant Kähler metric $h = h^{TX}$ on X which is invariant under complex conjugation.

A hermitian G -bundle on \mathcal{X} is a pair (\mathcal{F}, f) , where \mathcal{F} is a locally free G - \mathcal{X} sheaf with the property that the induced holomorphic vector bundle $\mathcal{F}_{\mathbf{C}}$ over X supports a G -invariant hermitian metric f , which is invariant with complex conjugation.

The complex lines $\det(\text{H}^\bullet(\text{R}\Gamma(\mathcal{X}, \mathcal{F})))_\phi$, for $\phi \in \widehat{G}$, carry metrics $f_{L^2, \phi}$ coming from the L^2 -metric of Hodge theory for the Dolbeault resolution. As per Section II in [B], the

$f_{L^2, \phi}$ can be transformed to equivariant Quillen metrics $f_{Q, \phi}$ for $\phi \in \widehat{G}$. One of the main objectives of this article is the study of the arithmetic classes

$$\chi(\mathrm{R}\Gamma(\mathcal{X}, \mathcal{F}), f_{Q, \bullet}) \text{ in } A(\mathbf{Z}[G]).$$

More generally, we shall also consider a bounded complex \mathcal{G}^\bullet of hermitian G -bundles on \mathcal{X} , with g^i denoting the hermitian form on \mathcal{G}^i . Then the g^\bullet induce metrics $g_{Q, \phi}^\bullet$ on the equivariant determinant of the hypercohomology of \mathcal{G}^\bullet , and so the arithmetic class

$$\chi(\mathrm{R}\Gamma(\mathcal{X}, \mathcal{G}^\bullet), g_{Q, \bullet}^\bullet) \text{ in } A(\mathbf{Z}[G])$$

is defined; explicitly, we may identify the equivariant determinant of $\det(\mathrm{H}^\bullet(\mathrm{R}\Gamma(\mathcal{X}, \mathcal{G}^\bullet)))_\phi$ with the product

$$\det(\mathrm{H}^\bullet(\mathrm{R}\Gamma(\mathcal{X}, \mathcal{G}^\bullet)))_\phi = \bigotimes_i \det(\mathrm{H}^\bullet(\mathrm{R}\Gamma(\mathcal{X}, \mathcal{G}^i)))_\phi^{(-1)^i}$$

and so

$$\chi(\mathrm{R}\Gamma(\mathcal{X}, \mathcal{G}^\bullet), g_{Q, \bullet}^\bullet) = \prod_i \chi(\mathrm{R}\Gamma(\mathcal{X}, \mathcal{G}^i), g_{Q, \bullet}^i)^{(-1)^i}.$$

In the sequel we shall write $g_{Q, \bullet}$ for the metrics on the equivariant determinant of hypercohomology induced by the $\{g_{Q, \bullet}^i\}$.

6.B 1-dimensional subschemes

In this sub-section we place ourselves in the situation described in 5.A. We now suppose further that \mathcal{X} and \mathcal{Y} are regular arithmetic varieties, with \mathcal{Y} being connected and having the property that at, any prime p of non-smooth reduction, $\mathcal{Y}_p^{\mathrm{red}}$ is a union of smooth irreducible components with strictly normal crossings and with each such component having multiplicity in \mathcal{Y}_p coprime to p . As noted previously, since G acts tamely on \mathcal{X} , the branch locus b of the cover \mathcal{X}/\mathcal{Y} is a vertical divisor. By hypothesis b is a divisor with strictly normal crossings. Here we consider an irreducible regular connected closed horizontal sub-scheme \mathcal{Z} of \mathcal{X} of dimension one; we may therefore write $\mathcal{Z} = \mathrm{Spec}(\mathcal{O}_N)$ for some ring of integers N , where G acts tamely on N . As previously we put $K = N^G$ consider $\mathcal{F} = \mathcal{O}_{\mathcal{Z}}$ and endow \mathcal{F} with the Hecke form h of 5.B.

Next we recall the Pfaffian divisor from Section 2 of [CEPT2]: for each symplectic character ψ of G , the Pfaffian divisor $\mathrm{Pf}(\mathcal{X}, \psi)$ is a divisor on \mathcal{Y} which is supported on the branch locus b . Let $\mathcal{W} = \pi(\mathcal{Z})$ so that \mathcal{W} is a closed sub-scheme of \mathcal{Y} . Throughout this sub-section we shall suppose that \mathcal{W} meets b transversely and at smooth points of b . As we

shall see in the next section, in practice we can often reduce to this situation by means of a moving lemma - subject to certain base extensions.

From Theorem 5.9 we know that $\chi^s(\mathcal{O}_{\mathcal{Z}}, \det h_{\bullet})$ is represented by the homomorphism $\tilde{\varepsilon}_{\infty}(K)^{-1} \text{Pf}(\mathcal{O}_N)^{-1} \times \delta_K$. Let $\{b_i\}$ denote the irreducible components of b ; let η_i denote the generic point of an irreducible component, B_i say, of $\pi^{-1}(b_i)$; let I_i denote the inertia group of η_i and recall that u_i denotes the augmentation character of I_i . From (2.1) in [CEPT2] we know that for $\psi \in R_G^s$

$$\text{Pf}(\mathcal{X}, \psi) = \frac{1}{2} \sum_i (\psi, \text{Ind}_{I_i}^G u_i) b_i. \quad (13)$$

A closed point \mathfrak{p} of \mathcal{W} (above p , say) is ramified in \mathcal{Z}/\mathcal{W} if and only if it is a point of intersection of \mathcal{W} and some b_i . Since we have assumed that \mathcal{W} intersects b transversely at smooth points of b , $I_{\mathfrak{p}}$ is a conjugate of I_i and recall that we denote the residue class degree of the point \mathfrak{p} by $f_{\mathfrak{p}}$. In the sequel for such a point \mathfrak{p} we write $n(\mathfrak{p}) = i$. By (5.8) for $\psi \in R_G^s$ we have

$$\text{Pf}_p(\mathcal{Z}, \psi) = \prod_{\mathfrak{p}} (-p)^{\frac{1}{2} f_{\mathfrak{p}} (\psi, \text{Ind}_{I_{n(\mathfrak{p})}}^G u_{n(\mathfrak{p})})}$$

where the product extends over all points of intersection of \mathcal{W} with the fibre of b above p . We therefore denote the right-hand expression by $\deg(\mathcal{W} \cdot \text{Pf}_p(\mathcal{X}, \psi))$, and we let $\deg(\mathcal{W} \cdot \text{Pf}(\mathcal{X}, \psi))$ denote the finite idele whose p -th component is $\deg(\mathcal{W} \cdot \text{Pf}_p(\mathcal{X}, \psi))$. (Note that almost all p -components are 1 and that the use of $-p$ in place of p means that of course we are using degree in a non-standard way.) Writing $\tilde{\varepsilon}_{\infty}(\mathcal{W})$ for $\tilde{\varepsilon}_{\infty}(K)$, we have now shown that the class $\chi^s(\mathcal{O}_{\mathcal{Z}}, \det h_{\bullet})$ is represented by the homomorphism

$$\tilde{\varepsilon}_{\infty}(\mathcal{W})^{-1} \cdot \deg(\mathcal{W} \cdot \text{Pf}(\mathcal{X}))^{-1} \times \delta_K.$$

Since the Dolbeault complex of a point is trivial, the equivariant Quillen metrics associated to the metrics h_{\bullet} are precisely the $\det(h_{\bullet})$ (cf. Definitions 2.1 and 2.2 in [B]). So finally we have now established the main result of this sub-section

Theorem 6.1 *The symplectic arithmetic class $\chi^s(\mathcal{O}_{\mathcal{Z}}, \det h_{Q\bullet})$ is represented by the homomorphism*

$$\tilde{\varepsilon}_{\infty}(\mathcal{W})^{-1} \deg(\mathcal{W} \cdot \text{Pf}(\mathcal{X}))^{-1} \times \delta_K.$$

6.C Invariance under passage to degree zero.

In this sub-section we establish a number of results concerning the independence, with respect to the choice of hermitian metric, of arithmetic classes after passage to degree zero

by the method described in 4.E. Recall that we denote the complexified generic fibre of \mathcal{X} by X .

Theorem 6.2 *Suppose that \mathcal{F} is a hermitian G -bundle on X and let f, f' be two G -invariant hermitian metrics on \mathcal{F} . Then there exists a positive real number c such that for each $\phi \in \widehat{G}$*

$$f_{Q,\phi} = c^{\phi(1)^2} f'_{Q,\phi}$$

and so

$$\widetilde{\chi}(\mathrm{R}\Gamma\mathcal{F}, f_{Q,\bullet}) = \widetilde{\chi}(\mathrm{R}\Gamma\mathcal{F}, f'_{Q,\bullet}).$$

PROOF. For each $\phi \in \widehat{G}$, let β_ϕ be the positive real number such that $\beta_\phi f_{Q,\phi} = f'_{Q,\phi}$. As previously we extend β to R_G by setting $\beta(\phi) = \beta_\phi^{1/\phi(1)}$, $\beta(\phi + \psi) = \beta(\phi)\beta(\psi)$ etc. In [B], Bismut considers the central function σ on G

$$\sigma = \sum_{\phi \in \widehat{G}} 2 \log(\beta_\phi) \phi(1)^{-1} \phi;$$

the Anomaly Formula in Theorem 2.5 of [B] shows that $\sigma(g)$ may be evaluated in terms of integrals over the fixed points of g . However, since G acts freely on X , for each $g \in G$, $g \neq 1_G$, the sub-variety of fixed points $X^g = \{x \in X(\mathbf{C}) \mid x^g = x\}$ is empty. Thus we immediately deduce that $\sigma(g) = 0$ whenever $g \neq 1_G$. This then shows that σ is a scalar multiple of the regular character and the result follows. \square

Next we consider the direct image of a hermitian bundle on a closed sub-scheme of a regular arithmetic variety \mathcal{X} . The formation of standard (i.e. non-hermitian) Euler characteristics respects closed immersions; however, this need not be the case for arithmetic classes, as the associated Quillen metrics may change. The precise variation in the arithmetic classes, that we wish to consider, was determined in Theorem 0.1 in [B].

We begin by considering a G -equivariant closed immersion $i : \mathcal{Z} \rightarrow \mathcal{X}$ of an arithmetic variety \mathcal{Z} which also supports a tame action by G . Let \mathcal{F} denote a locally free G - \mathcal{Z} sheaf. Since \mathcal{X} is regular, we may resolve $i_*\mathcal{F}$ by a bounded complex \mathcal{G}^\bullet of locally free coherent G - \mathcal{X} modules. We then have natural isomorphisms in the derived category of $\mathbf{Z}[G]$ -modules

$$\mathrm{R}\Gamma(\mathcal{Z}, \mathcal{F}) \cong \mathrm{R}\Gamma(\mathcal{X}, i_*\mathcal{F}) \cong \mathrm{R}\Gamma(\mathcal{X}, \mathcal{G}^\bullet) \tag{14}$$

and hence, for each $\phi \in \widehat{G}$, we obtain isomorphisms

$$\sigma_\phi : \det \mathrm{H}^\bullet(\mathrm{R}\Gamma(\mathcal{X}, \mathcal{G}^\bullet))_\phi \cong \det \mathrm{H}^\bullet(\mathrm{R}\Gamma(\mathcal{Z}, \mathcal{F}))_\phi. \tag{15}$$

In order to describe the relevant metrics that we wish to place on these determinants of cohomology, we need some further notation. Let $Z = \mathcal{Z}_{\mathbf{C}}$ and let TZ denote the tangent bundle of Z . We let h^{TZ} denote the restriction of h to TZ . Let $N_{Z|X}$ denote the normal bundle to Z in X and let $h^{N_{Z|X}}$ be the metric on $N_{Z|X}$ induced by h . Let f denote a given G -invariant metric on \mathcal{F} ; we then endow each term \mathcal{G}^i of \mathcal{G}^\bullet with a G -invariant hermitian metric g^i in such a way that the metrics $\{g^i\}$ satisfy Bismut's Condition A with respect to $h^{N_{Z|X}}$ and f .

We now wish to compare the arithmetic classes $\chi(\mathrm{R}\Gamma(\mathcal{Z}, F), f_{Q\bullet})$ and $\chi(\mathrm{R}\Gamma(\mathcal{X}, \mathcal{G}^\bullet), g_{Q\bullet})$.

Let α_ϕ be the unique positive real number such that under the isomorphism σ_ϕ of (15)

$$\sigma_\phi^*(f_{Q,\phi}) = \alpha_\phi g_{Q,\phi}.$$

Then by Proposition 5.4 we see that the arithmetic class

$$\chi(\mathrm{R}\Gamma(\mathcal{X}, \mathcal{G}^\bullet), g_{Q\bullet}) \cdot \chi(\mathrm{R}\Gamma(\mathcal{Z}, \mathcal{F}), f_{Q\bullet})^{-1}$$

is represented by the homomorphism $1 \times \alpha^{-1} \in \mathrm{Hom}_{\Omega_{\mathbf{Q}}}(R_G^s, J_f) \times \mathrm{Hom}(R_G^s, \mathbf{R}_{>0})$ which maps the character ϕ to $1 \times \alpha_\phi^{-1/\phi(1)}$ (so that of course $\alpha(\phi) = \alpha_\phi^{1/\phi(1)}$).

Theorem 6.3 *With the above notation and hypotheses there is a positive real number b such that for each $\phi \in \widehat{G}$, $\alpha_\phi = b^{\phi(1)^2}$ and so*

$$\tilde{\chi}(\mathrm{R}\Gamma(\mathcal{X}, \mathcal{G}^\bullet), g_{Q\bullet}) = \tilde{\chi}(\mathrm{R}\Gamma(\mathcal{Z}, \mathcal{F}), f_{Q\bullet}).$$

PROOF. In Theorem 0.1 in [B] Bismut considers the central function τ

$$\tau = \sum_{\phi \in \widehat{G}} 2 \log(\alpha_\phi) \phi(1)^{-1} \phi.$$

and shows that $\tau(g)$ may be evaluated in terms of integrals over the fixed points of g . As in the proof of 6.2 we deduce that $\tau(g) = 0$ whenever $g \neq 1_G$. This then shows that τ is again a scalar multiple of the regular character and the result follows. \square

We now interpret the above results in terms of arithmetic classes.

Proposition 6.4 *Let (\mathcal{F}_j, f_j) for $j = 1, \dots, n$ and (\mathcal{G}_k, g_k) for $k = 1, \dots, m$ be hermitian bundles on closed G -subschemes $i_j : \mathcal{Z}_j \rightarrow \mathcal{X}$, $i_k : \mathcal{W}_k \rightarrow \mathcal{X}$ such that*

$$\sum_j [i_{j*} \mathcal{F}_j] = \sum_k [i_{k*} \mathcal{G}_k] \quad \text{in } K_0(G, \mathcal{X}).$$

Then there is an equality of classes in $A(\mathbf{Z}[G])$

$$\prod_j \tilde{\chi}(\mathrm{R}\Gamma(\mathcal{Z}_j, \mathcal{F}_j), f_{j,Q\bullet}) = \prod_k \tilde{\chi}(\mathrm{R}\Gamma(\mathcal{W}_k, \mathcal{G}_k), g_{k,Q\bullet}) .$$

PROOF. We first choose resolutions by locally free G - \mathcal{X} sheaves

$$\mathcal{A}_j^\bullet \rightarrow i_{j*}\mathcal{F}_j, \quad \mathcal{B}_k^\bullet \rightarrow i_{k*}\mathcal{G}_k.$$

From the definition of $K_0(G, \mathcal{X})$, we can find locally free G - \mathcal{X} sheaves $D_{a,b}$, $E_{c,d}$ and an isomorphism, which we henceforth treat as an equality,

$$\begin{aligned} & \oplus_b D_{2,b} \oplus_d E_{1,d} \oplus_i E_{3,d} \oplus_{j,a \text{ even}} \mathcal{A}_j^a \oplus_{k,b \text{ odd}} \mathcal{B}_k^b \\ &= \oplus_d E_{2,d} \oplus_b D_{1,b} \oplus_j D_{3,b} \oplus_{j,a \text{ odd}} \mathcal{A}_j^a \oplus_{k,b \text{ even}} \mathcal{B}_k^b \end{aligned} \quad (16)$$

where the G - \mathcal{X} sheaves $D_{a,b}$, $E_{c,d}$ fit into exact sequences

$$0 \rightarrow E_{1,d} \rightarrow E_{2,d} \rightarrow E_{3,d} \rightarrow 0$$

$$0 \rightarrow D_{1,b} \rightarrow D_{2,b} \rightarrow D_{3,b} \rightarrow 0.$$

We then endow the sheaves $E_{3,d}$ and $D_{3,b}$ with arbitrary G -invariant metrics $\xi_{3,d}$ and $\eta_{3,b}$; we then choose G -invariant metrics $\xi_{1,d}$, $\xi_{2,d}$, $\eta_{1,b}$, $\eta_{2,b}$ on $E_{1,d}$, $E_{2,d}$, $D_{1,b}$, $D_{2,b}$ satisfying Condition A as above, so that by Theorem 6.3:

$$\tilde{\chi}(\mathrm{R}\Gamma D_{1,b}, \xi_{1,d,Q}) \cdot \tilde{\chi}(\mathrm{R}\Gamma D_{3,b}, \xi_{3,d,Q}) = \tilde{\chi}(\mathrm{R}\Gamma D_{2,b}, \xi_{2,d,Q}) \text{ etc.}$$

We then endow the sheaves $\mathcal{A}_j^a, \mathcal{B}_k^b$ with G -invariant metrics α_j^a, β_k^b satisfying Condition A, so that by Theorem 6.3

$$\tilde{\chi}(\mathrm{R}\Gamma \mathcal{A}_j^\bullet, \alpha_{j,Q\bullet}) = \tilde{\chi}(\mathrm{R}\Gamma(i_{j*}\mathcal{F}_j), f_{j,Q\bullet}) \text{ etc.}$$

The desired equality then follows from (16) and Theorem 6.2. \square

7 Logarithmic Differentials

In this section we consider the Arakelov-Euler characteristic associated to the logarithmic de Rham complex of an arithmetic variety \mathcal{X} with fibral dimension d . We begin by relating this class to an arithmetic class associated to the top Chern class of the logarithmic differentials of \mathcal{X} . After allowing for various innocuous base field extensions, we shall use the moving

techniques of [CPT1] to express this top Chern class as a difference of two horizontal 1-cycles together with a relatively innocuous fibral term. We shall then be able to use the results of 5.B to show that the arithmetic class associated to the logarithmic de Rham complex of \mathcal{X} has the remarkable property of characterising symplectic ε_0 -constants of \mathcal{X} .

Recall that we have fixed a Kähler metric h on the tangent bundle of $X = \mathcal{X}(\mathbf{C})$ and we let h^D denote the dual metric induced on the cotangent bundle of X .

In this section we again suppose, that \mathcal{X} and \mathcal{Y} are as described in 6.B. Let S denote a finite set of prime numbers which contains all the primes which support the branch locus, together with all primes p where the fibre \mathcal{Y}_p fails to be smooth. We put $S' = S \cup \{\infty\}$.

Let $\chi(\mathcal{Y}_{\mathbf{Q}}) = \chi(\mathcal{Y}(\mathbf{C}))$ denote the Euler characteristic of the generic fibre of \mathcal{Y} . Note that in all cases $d \cdot \chi(\mathcal{Y}_{\mathbf{Q}})$ is an even integer, so that we may define $\xi_S : R_G \rightarrow \mathbf{Q}^\times$ by the rule

$$\xi_S(\theta) = \prod_{p \in S} p^{-\theta(1) \cdot d \cdot \chi(\mathcal{Y}_{\mathbf{Q}})/2}.$$

Let $\Omega_{\mathcal{Y}/\mathbf{Z}}^1(\log \mathcal{Y}_S^{\text{red}} / \log S)$ denote the sheaf of degree one relative logarithmic differentials with respect to the morphism $(\mathcal{Y}, \mathcal{Y}_S^{\text{red}}) \rightarrow (\text{Spec}(\mathbf{Z}), S)$ of schemes with log-structures (see [K]). Under our hypotheses $\Omega_{\mathcal{Y}/\mathbf{Z}}^1(\log \mathcal{Y}_S^{\text{red}} / \log S)$ is a locally free \mathcal{Y} -sheaf of rank d , and furthermore the cover \mathcal{X}/\mathcal{Y} is log-étale, so that

$$\Omega_{\mathcal{X}/\mathbf{Z}}^1(\log \mathcal{X}_S^{\text{red}} / \log S) = \pi^* \Omega_{\mathcal{Y}/\mathbf{Z}}^1(\log \mathcal{Y}_S^{\text{red}} / \log S). \quad (17)$$

The main goal of this section is the study of the arithmetic class (see 6.A)

$$\begin{aligned} \mathfrak{c} &= \chi(\text{R}\Gamma(\wedge^\bullet \Omega_{\mathcal{X}/\mathbf{Z}}^1(\log \mathcal{X}_S^{\text{red}} / \log S), \wedge^\bullet h_Q^D)) \\ &= \prod_{i=0}^d \chi(\text{R}\Gamma(\wedge^i \Omega_{\mathcal{X}/\mathbf{Z}}^1(\log \mathcal{X}_S^{\text{red}} / \log S), \wedge^i h_Q^D)^{(-1)^i}). \end{aligned}$$

To explain our main result we need to introduce some notation on ε_0 -constants. For a more detailed account see Sect. 4 in [CEPT2] and Sect. 2 and Sect. 5 in [CEPT1]. For a given prime number p , we choose a prime number $l = l_p$ which is different from p and we fix a field embedding $\mathbf{Q}_l \rightarrow \mathbf{C}$; then, following the procedure of Sect. 8 in [D], each of the étale cohomology groups $H_{\text{ét}}^i(\mathcal{X} \times \bar{\mathbf{Q}}_p, \mathbf{Q}_l)$ for $0 \leq i \leq 2d$, affords a continuous complex representation of the local Weil-Deligne group. Thus, after choosing both an additive character ψ_p of \mathbf{Q}_p and a Haar measure dx_p of \mathbf{Q}_p , for each complex character θ of G the complex number $\varepsilon_{0,p}(\mathcal{Y}, \theta, \psi_p, dx_p, l_p)$ is defined. (For a representation V of G with character θ this term was denoted $\varepsilon_{p,0}(X \otimes_G V, \psi_p, dx_p, l)$ in 2.4 of [CEPT1].) Setting $\tilde{\varepsilon}_{0,p}(\mathcal{Y}, \theta, \psi_p, dx_p, l_p) = \varepsilon_{0,p}(\mathcal{Y}, \theta - \theta(1) \cdot 1, \psi_p, dx_p, l_p)$, by Corollary 1 to Theorem 1

in [CEPT2] we know that when θ is symplectic, $\tilde{\varepsilon}_{0,p}(\mathcal{Y}, \theta, \psi_p, dx_p, l_p)$ is a non-zero rational number, which is independent of choices, and $\theta \mapsto \tilde{\varepsilon}_{0,p}(\mathcal{Y}, \theta)$ defines an element

$$\tilde{\varepsilon}_{0,p}^s(\mathcal{Y}) \in \text{Hom}_{\Omega}(R_G^s, \mathbf{Q}^{\times}).$$

In the case where \mathcal{X} is the spectrum of a ring of integers \mathcal{O}_N of a number field N and $K = N^G$, we shall write $\varepsilon_{0,p}(K)$ for $\varepsilon_{0,p}(\mathcal{Y})$.

Analogously, for the Archimedean prime ∞ of \mathbf{Q} , Deligne provides a definition for $\varepsilon_{\infty}(\mathcal{Y})$ and from 5.5.2 and 5.4.1 in [CEPT1] we recall that

$$\tilde{\varepsilon}_{\infty}^s(\mathcal{Y}) \in \text{Hom}_{\Omega}(R_G^s, \pm 1).$$

For $\phi \in R_G^s$ almost all $\tilde{\varepsilon}_{0,v}^s(\mathcal{Y}, \phi)$ are equal to 1; the global $\tilde{\varepsilon}_0$ -constant of ϕ is

$$\tilde{\varepsilon}_0^s(\mathcal{Y}, \phi) = \prod_v \tilde{\varepsilon}_{0,v}^s(\mathcal{Y}, \phi)$$

and we define

$$\varepsilon_{0,S}^s(\mathcal{Y}, \phi) = \tilde{\varepsilon}_0^s(\mathcal{Y}, \phi) \prod_{v \in S'} \varepsilon_{0,v}(\mathcal{Y}, \phi(1)).$$

The main result of this section is

Theorem 7.1 *The arithmetic class \mathfrak{c}^s lies in the group of rational classes $R_T^s(\mathbf{Z}[G])$ and*

$$\theta(\mathfrak{c}^s) = \xi_S^s \cdot \varepsilon_{0,S}^s(\mathcal{Y})^{-1}.$$

By way of preparation for the proof of Theorem 7.1, we shall initially work with an arbitrary locally free \mathcal{Y} -sheaf \mathcal{E} ; only towards the end of the section shall we need to specialise to the case where $\mathcal{E} = \Omega_{\mathcal{Y}/\mathbf{Z}}^1(\log \mathcal{Y}_S^{\text{red}} / \log S)$. Throughout this section we adopt the notation and hypotheses of [CPT1]. For $i \geq 0$ let $c^i(\mathcal{E}) = \gamma^i(\mathcal{E} - \text{rk}(\mathcal{E}))$ which lies in $F^i K_0(\mathcal{Y})$, the i -th component of the Grothendieck γ -filtration. We define $\underline{c}^i(\mathcal{E})$ to be the class

$$\underline{c}^i(\mathcal{E}) \equiv c^i(\mathcal{E}) \bmod F^{i+1} K_0(\mathcal{Y}).$$

Lemma 7.2 *Let \mathcal{E} be as above, let \mathcal{L} denote an arbitrary line bundle on \mathcal{Y} and suppose that n_0 is a given negative integer. Then there exists an integer $n_1 \leq n_0$ and integers l_n for $n_1 \leq n \leq n_0$, which depend only on $\text{rk}(\mathcal{E})$, such that for all $i \geq 0$*

$$\underline{c}^i(\mathcal{E}) \equiv \sum_{n=n_1}^{n_0} l_n \underline{c}^i(\mathcal{E} \otimes \mathcal{L}^n) \bmod F^{i+1} K_0(\mathcal{Y}). \quad (18)$$

PROOF. This is Lemma 5.3 in [CPT1] with \mathcal{L} replaced by \mathcal{L}^{-1} . \square

Definition 7.3 Let m be a given positive integer. We shall call a finite Galois extension M of \mathbf{Q} *harmless for m* , if M/\mathbf{Q} is non-ramified at S and if the extension degree $[M : \mathbf{Q}]$ is congruent to 1 mod m .

Remark From 9.1.2 in [CEPT1] we know that we can construct harmless for m extensions whose residue class fields over S are arbitrarily large.

If m is a positive integer and if M is harmless for m , let $e : \text{Spec}(\mathcal{O}_M) \rightarrow \text{Spec}(\mathbf{Z})$ be the structure morphism, write \mathcal{Y}' for the base extension $\mathcal{Y} \times_{\mathbf{Z}} \mathcal{O}_M$, and \mathcal{E}' for the pullback of \mathcal{E} to \mathcal{Y}' .

Suppose now that an integer m is given and that \mathcal{E} has rank d ; for an integer n we put $\mathcal{E}(n) = \mathcal{E} \otimes_{\mathcal{O}_{\mathcal{Y}}}(n)$. From 5.1 in [CPT1] we know that we can find a negative integer n_0 with the following property: let n_1 be an integer chosen as in Lemma 7.2 with respect to the negative integer n_0 ; then for each n , $n_1 \leq n \leq n_0$, there is an open subset U_n of $\text{Spec}(\mathbf{Z})$, which contains S , an extension M which is harmless for m , and a (possibly non-effective) 1-cycle D'_n on \mathcal{Y}' whose irreducible components are horizontal and meet b transversely and at points which are smooth points of both D'_n and b , such that

$$c^d(\mathcal{E}'(n))|_{U_n} = [O_{D'_n \times U_n}] + T_n \text{ in } K_0(\mathcal{Y}' \times U_n)$$

where T_n is supported on closed points. With the notation of Lemma 7.2, we set $U = \bigcap_{n=n_1}^{n_0} U_n$, so that for all n , $n_1 \leq n \leq n_0$ we have

$$c^d(\mathcal{E}'(n))|_U = [O_{D'_n \times U}] + T_n|_U \text{ in } K_0(\mathcal{Y}' \times U).$$

Pushing forward by e and using the fact that \mathcal{O}_M is free over \mathbf{Z} , we get

$$[M : \mathbf{Q}] \cdot c^d(\mathcal{E}(n))|_U = [O_{e_* D'_n \times U}] + e_*(T_n|_U) \text{ in } K_0(\mathcal{Y} \times U). \quad (19)$$

In the sequel we work with a chosen such extension M . We write $\widehat{\pi^* D'_n}$ for the normalisation of $\pi^* D'_n$ and we endow both $e_* \pi^* D'_n$ and $\widehat{e_* \pi^* D'_n}$ with the Hecke form h_n of 6.B; we denote their arithmetic classes by $\chi(e_* \pi^* D'_n, \det h_{n\bullet})$ and $\tilde{\chi}(\widehat{e_* \pi^* D'_n}, \det h_{n\bullet})$. Proposition 5.7, together with Lemma 7.4 below, shows that the two resulting classes coincide after passage to degree zero

$$\tilde{\chi}(e_* \pi^* D'_n, \det h_{n\bullet}) = \tilde{\chi}(\widehat{e_* \pi^* D'_n}, \det h_{n\bullet}). \quad (20)$$

Lemma 7.4 Suppose that \mathcal{F} is a coherent \mathcal{Y} -sheaf which is supported on a single prime p and suppose: either that $p \notin S$; or that, if $p \in S$, then \mathcal{F} is supported over a finite number of points of \mathcal{Y} . Then $f_{p*}(\pi^* \mathcal{F})$ is a free class in $K_0(\mathbf{F}_p[G])$.

PROOF. Let $h : \mathcal{Y} \rightarrow \operatorname{Spec}(\mathbf{Z})$ denote the structure morphism of \mathcal{Y} and suppose first that \mathcal{F} is the coherent \mathcal{Y} -sheaf given by the structure sheaf of a closed point of \mathcal{Y} . As $f_* = h_*\pi_*$ and $\pi_*\pi^*\mathcal{F} = \mathcal{F} \otimes_{\mathcal{O}_{\mathcal{Y}}} \pi_*\mathcal{O}_{\mathcal{X}}$ in $G_0(G, \mathcal{Y})$, the result follows readily from the normal basis theorem.

Suppose now that $p \notin S$. For a p -regular element $g \in G, g \neq 1, \mathcal{X}_p^g = \emptyset$, since G acts freely away from S . Thus by the Lefschetz-Riemann-Roch theorem, we know that the Brauer trace of g on $f_{p*}(\mathcal{F})$ is zero; hence we may conclude that $f_{p*}(\mathcal{F})$ is a free class. \square

Recall that $\tilde{\mathfrak{c}}$ denotes the arithmetic class obtained from \mathfrak{c} by passage to degree zero, as per 4.E. As an intermediate step towards proving Theorem 7.1, we first show that the result holds in degree zero:

Theorem 7.5 *The arithmetic class $\tilde{\mathfrak{c}}^s$ lies in the group of rational classes $R_T^s(\mathbf{Z}[G])$ and*

$$\theta(\tilde{\mathfrak{c}}^s) = \tilde{\varepsilon}_0^s(\mathcal{Y})^{-1}.$$

PROOF. We apply the above work where we now take $\mathcal{E} = \Omega_{\mathcal{Y}/\mathbf{Z}}^1(\log \mathcal{Y}_S^{\text{red}}/\log S)$ and where we take n_0 sufficiently small and negative to guarantee that $\mathcal{E}^D(-n)$ has a regular section for all $n \leq n_0$. Recall that $\pi^*\mathcal{E}$ is endowed with the metric h^D , the dual of the Kähler metric; we endow $\pi^*\mathcal{O}_{\mathcal{Y}}(n)$ with a chosen G -invariant metric ν_n .

By 7.2 together with 7.4 and Proposition 6.4, we know that

$$\tilde{\chi}(\operatorname{R}\Gamma(\wedge^\bullet \pi^*\mathcal{E}), (\wedge^\bullet h^D)_Q) = \prod_{n=n_0}^{n_1} \tilde{\chi}(\operatorname{R}\Gamma(\wedge^\bullet \pi^*\mathcal{E}(n)), (\wedge^\bullet h^D \otimes \nu_n)_Q)^{l_n}. \quad (21)$$

Let \mathcal{W}_n denote the closed one dimensional sub-scheme of \mathcal{Y} cut out by the regular section of $\mathcal{E}^D(-n)$ and put $\mathcal{Z}_n = \pi^*\mathcal{W}_n$; so that we have the Koszul quasi-isomorphism

$$\wedge^\bullet \mathcal{E}(n) \rightarrow \mathcal{O}_{\mathcal{W}_n}.$$

By 6.4 we know that

$$\tilde{\chi}(\operatorname{R}\Gamma(\wedge^\bullet \pi^*\mathcal{E}(n)), (\wedge^\bullet h^D \otimes \nu_n)_Q) = \tilde{\chi}(\operatorname{R}\Gamma \mathcal{O}_{\mathcal{Z}_n}, j_n) \quad (22)$$

where j_n denotes the Hecke metric on $\mathcal{O}_{\mathcal{Z}_n}$. Next observe that $[\wedge^\bullet \pi^*\mathcal{E}(n)] = (-1)^d c^d(\pi^*\mathcal{E}(n))$, and also by 7.4 we know that $\tilde{\chi}(e_*\pi^*T_n, \pi^*|-|) = 0$; hence by Proposition 6.4, together with (19), (20), (21) and (22), we may conclude that

$$\prod_{n=n_1}^{n_0} \tilde{\chi}(\operatorname{R}\Gamma \mathcal{O}_{\mathcal{Z}_n}, j_n)^{l_n[M:\mathbf{Q}]} = \prod_{n=n_1}^{n_0} \tilde{\chi}(e_*\pi^*D'_n, \det h_{n\bullet})^{(-1)^d l_n}$$

$$= \prod_{n=n_1}^{n_0} \widetilde{\chi}(e_* \pi^* \widehat{D'_n}, \det h_{n\bullet})^{(-1)^{d_{l_n}}}. \quad (23)$$

Let $\mathcal{C} = (-1)^d c^d(\mathcal{E})$ and consider the restriction of \mathcal{C} to an irreducible component b_i of b over p ; in this way we obtain a punctual virtual sheaf whose length we denote by n_i . If, for $\psi \in R_G^s$, we have $\text{Pf}_p(\mathcal{X}, \psi) = \sum_i q_i b_i$ (see (13)), then we may define $\deg(\mathcal{C} \cdot \text{Pf}(\mathcal{X}))(\psi) \in J_f$ to be the idele whose component at primes over p is $(-p)^{\sum_i q_i n_i}$. We then use this construction to define the symplectic arithmetic class $\mathfrak{h} \in A_T^s(\mathbf{Z}[G])$ to be that class which is represented by the homomorphism

$$(\widetilde{\varepsilon}_\infty^s(\mathcal{Y}) \cdot \deg(\mathcal{C} \cdot \text{Pf}(\mathcal{X}))) \times 1.$$

In fact, from Theorem 1 in [CEPT2], we know that \mathfrak{h} is a rational class and that moreover

$$\theta(\mathfrak{h}) = \widetilde{\varepsilon}_\infty^s(\mathcal{Y}) \prod_{p < \infty} \widetilde{\varepsilon_{0,p}^s}(\mathcal{Y}) = \widetilde{\varepsilon}_0^s(\mathcal{Y}). \quad (24)$$

By Theorems 5.9 and 6.1, the left-hand arithmetic class in (23) above is represented by the character function given on characters of degree zero by

$$\begin{aligned} & \prod_{n=n_1}^{n_0} \left(\widetilde{\varepsilon}_\infty^s(\pi^* \widehat{D'_n}) \deg(\pi^* \widehat{D'_n} \cdot \text{Pf}(\mathcal{X}')) \right)^{-(-1)^{d_{l_n}}} \times 1 \\ &= \prod_{n=n_1}^{n_0} \left(\widetilde{\varepsilon}_\infty^s(e_* \pi^* D'_n) \deg(e_* \pi^* D'_n \cdot \text{Pf}(\mathcal{X})) \right)^{-(-1)^{d_{l_n}}} \times 1. \end{aligned} \quad (25)$$

By (18) and (19) together with Theorem 5.5.2 in [CEPT1] we know that

$$\prod_{n=n_1}^{n_0} \widetilde{\varepsilon}_\infty^s(e_* \pi^* D'_n)^{l_n} = \widetilde{\varepsilon}_\infty^s(\mathcal{Y})^{[M:\mathbf{Q}]}.$$

Again by (18) and (19)

$$\prod_{n=n_1}^{n_0} (\deg(e_* \pi^* D'_n \cdot \text{Pf}(\mathcal{X})))^{l_n} = \deg((-1)^d \mathcal{C} \cdot \text{Pf}(\mathcal{X}))^{[M:\mathbf{Q}]}. \quad (26)$$

By (21), (22) and (23) we know that

$$\widetilde{\chi}(\text{R}\Gamma(\wedge^\bullet \pi^* \mathcal{E}), (\wedge^\bullet h^D)_Q)^{[M:\mathbf{Q}]} = \prod_{n=n_1}^{n_0} \widetilde{\chi}(e_* \pi^* \widehat{D'_n}, \det h_{n\bullet})^{(-1)^{d_{l_n}}}$$

and by the above work the right-hand class is represented by the same homomorphism as $\mathfrak{h}^{-[M:\mathbf{Q}]}$. Thus, by varying M , we see that

$$\widetilde{\chi}(\text{R}\Gamma(\wedge^\bullet \pi^* \mathcal{E}), (\wedge^\bullet h^D)_Q) = \mathfrak{h}^{-1}$$

and so by (24)

$$\theta \left(\widetilde{\chi}(\mathrm{R}\Gamma(\wedge^{\bullet} \pi^* \mathcal{E}), (\wedge^{\bullet} h^D)_Q) \right) = \widetilde{\varepsilon}_0^s(\mathcal{Y})^{-1}. \quad \square$$

Before embarking on the proof of Theorem 7.1, we first need a number of preliminary results.

Lemma 7.6 (a) *For a coherent G - \mathcal{X} sheaf \mathcal{F} there is a quasi-isomorphism of $\mathbf{Z}[G]$ -complexes*

$$(\mathrm{R}\Gamma \mathcal{F})^G \cong \mathrm{R}\Gamma(\mathcal{F}^G).$$

(b) *If (\mathcal{F}, f) is a hermitian G -bundle on \mathcal{X} , then there is an equality in $A(\mathbf{Z}[G])$*

$$\chi \left((\mathrm{R}\Gamma \mathcal{F})^G, f_{Q,1} \right) = \chi \left(\mathrm{R}\Gamma(\mathcal{F}^G), (f^G)_Q \right).$$

PROOF. Part (a) follows at once on expressing $\mathrm{R}\Gamma \mathcal{F}$ and $\mathrm{R}\Gamma(\mathcal{F}^G)$ in terms of Čech complexes for a given affine cover of \mathcal{Y} (which pulls back to an affine cover of \mathcal{X} , since \mathcal{X}/\mathcal{Y} is finite) and then taking G -invariants of the first complex. Part (b) is then immediate since $f_{Q,1}$ (the Quillen metric for the trivial character) is constructed by forming the Quillen metric associated to the restriction of f to the trivial isotypical component of $\mathcal{F}_{\mathbf{C}}$, namely $(\mathcal{F}_{\mathbf{C}})^G$. See II.a in [B] for further details. \square

Next we note the following elementary result from 3.2:

Lemma 7.7 *When G is the trivial group, then there is an isomorphism $\gamma : A(\mathbf{Z}) \rightarrow \mathbf{R}_{>0}$, (which coincides with the degree map on p. 162 of [S]). Furthermore, if a class $\mathfrak{e} \in A(\mathbf{Z})$ has $\gamma(\mathfrak{e})^2 \in \mathbf{Q}_{>0}$, then the symplectic class \mathfrak{e}^s is a rational class and $\theta(\mathfrak{e}^s) = \gamma(\mathfrak{e})^2$.*

PROOF. For a rational finite idele $j \in J_{\mathbf{Q},f}$ we write $c(j)$ for the positive rational number which generates the fractional \mathbf{Z} -ideal given by the content of j . The first part of the lemma then follows from 3.2 on noting that the map from $J_{\mathbf{Q},f} \times \mathbf{R}_{>0} \rightarrow \mathbf{R}_{>0}$ given by mapping $(j, r) \mapsto c(j) r^{-1}$ has kernel $(\widehat{\mathbf{Z}}^{\times} \times 1) \cdot \Delta(\mathbf{Q}^{\times})$. To show the second part of the lemma we first note that the class \mathfrak{e} is represented by $1 \times \gamma(\mathfrak{e})^{-1}$ and that the symplectic characters of the trivial group are the even multiples of the trivial character. It therefore follows that \mathfrak{e}^s is represented by $1 \times \gamma(\mathfrak{e})^{-2}$, which has the same class in $A_T^s(\mathbf{Z})$ as $\gamma(\mathfrak{e})^2 \times 1$. \square

We denote by $\mathcal{Y}_S^{\mathrm{red}}$ the disjoint union of the reduced fibres of \mathcal{Y} over $p \in S$. Let \mathcal{Y}_i , for $i \in \mathcal{I}$, denote the irreducible components of $\mathcal{Y}_S^{\mathrm{red}}$, so that

$$\mathcal{Y}_S^{\mathrm{red}} = \cup_{i \in \mathcal{I}} \mathcal{Y}_i.$$

Let p_i denote the prime which supports \mathcal{Y}_i and let $\chi_c(\mathcal{Y}_i^*)$ denote the ℓ -adic Euler characteristic with compact supports of $\mathcal{Y}_i^* = \mathcal{Y}_i - \cup_{j \neq i} \mathcal{Y}_j$, the non-singular part of \mathcal{Y}_i .

Thanks to Theorem 7.5, in order to prove Theorem 7.1, we need only show that, with the notation of 4.2, \mathfrak{c}_0^s is a rational class and that

$$\theta(\mathfrak{c}_0^s) = \xi_S(2) \prod_{v \in S'} \varepsilon_{0,v}(\mathcal{Y}, 2)^{-1}.$$

Therefore, by 7.7, it will suffice to show that

$$\gamma(\mathfrak{c}_0)^2 = \xi_S(2) \prod_{v \in S'} \varepsilon_{0,v}(\mathcal{Y}, 2)^{-1} \in \mathbf{Q}_{>0}.$$

From (17), we know that for all non-negative j

$$\left(\bigwedge^j \Omega_{\mathcal{X}/\mathbf{Z}}^1(\log \mathcal{X}_S^{\text{red}} / \log S) \right)^G = \bigwedge^j \Omega_{\mathcal{Y}/\mathbf{Z}}^1(\log \mathcal{Y}_S^{\text{red}} / \log S)$$

hence

$$\mathfrak{c}_0 = \prod_{j=0}^d \chi \left(\text{R}\Gamma \wedge^j \Omega_{\mathcal{Y}/\mathbf{Z}}^1(\log \mathcal{Y}_S^{\text{red}} / \log S), \wedge^j h_{Q,1}^D \right)^{(-1)^j}$$

which we write more succinctly as

$$\mathfrak{c}_0 = \chi \left(\text{R}\Gamma \wedge^\bullet \Omega_{\mathcal{Y}/\mathbf{Z}}^1(\log \mathcal{Y}_S^{\text{red}} / \log S), \wedge^\bullet h_{Q,1}^D \right).$$

We therefore see that it is enough to show the following two results:

Theorem 7.8

$$\gamma \circ \chi \left(\text{R}\Gamma \wedge^\bullet \Omega_{\mathcal{Y}/\mathbf{Z}}^1(\log \mathcal{Y}_S^{\text{red}} / \log S), \wedge^\bullet h_{Q,1}^D \right) = \prod_{i \in \mathcal{I}} p_i^{-(m_i-1)\chi_c(\mathcal{Y}_i^*)}$$

for any Kähler metric j on the complex tangent bundle TY .

Remark In fact this result can also easily be proved using the arithmetic Riemann-Roch Theorem of Gillet and Soulé; this alternative approach to the calculation of Arakelov-Euler characteristics is explored in [CPT2]; here, however, we shall provide a direct proof, which is due to Bismut and which was shown to us by C. Soulé.

Theorem 7.9 $\varepsilon_{0,S}(\mathcal{Y}, 2) = \xi_S(2) \prod_{i \in \mathcal{I}} p_i^{(m_i-1)\chi_c(\mathcal{Y}_i^*)} \in \mathbf{Q}_{>0}.$

We begin by proving Theorem 7.9. For a place v of \mathbf{Q} we calculate the $\varepsilon_{0,v}$ -constants with respect to the standard Haar measures dx_v of \mathbf{Z}_v and with respect to the Tate-Iwasawa additive character ψ_v of \mathbf{Q}_v (see [Ta] p. 316-319).

We first consider the case of a finite prime p . From Theorem 2 in [Sa] we know that

$$\varepsilon_{0,p}(2, \mathcal{Y}, \psi_p \circ p^{-1}, p dx_p) = \pm \prod_{p_i=p} p^{(m_i-1)\chi_c^*(\mathcal{Y}_i)}.$$

Thus by the standard transformation formulae for ε -constants (see 5.3 and 5.4 in [D])

$$\varepsilon_{0,p}(2, \mathcal{Y}, \psi_p, dx_p) = \pm \sigma^2(p) \prod_{p_i=p} p_i^{(m_i-1)\chi_c^*(\mathcal{Y}_i)}$$

where σ denotes the determinant of the motive of $\mathcal{X} \otimes_G V$ and where V denotes the trivial representation of G . From Proposition 2.2.1.a,c in [CEPT1] we know that $\sigma^2(p) = p^{-d\chi(Y_{\mathbf{Q}})}$.

Next, we consider the archimedean prime $v = \infty$. From Lemma 5.1.1 in [CEPT1], we know that $\varepsilon_{0,\infty}(2, \mathcal{Y}, -\psi_{\infty}, dx_{\infty}) = \pm 1$.

To show that $\varepsilon_{0,S}(\mathcal{Y}, 2)$ is positive, note that from (2.2) in 2.4 of [CEPT1] we know that in all cases

$$\text{sign}(\varepsilon_{0,v}(\mathcal{Y}, 2)) = \det(\sigma)(-1_v).$$

Thus by global reciprocity $1 = \prod_{v \in S'} \det(\sigma)(-1_v)$ and so we have indeed now shown that $\varepsilon_{0,S}(\mathcal{Y}, 2)$ is a positive rational number. \square

Prior to proving Theorem 7.8, we note that we have:

Lemma 7.10 *Writing $\omega_{\mathcal{Y}/\mathbf{Z}}$ for the canonical sheaf of \mathcal{Y}/\mathbf{Z} , there is a natural isomorphism between $\wedge^d \Omega_{\mathcal{Y}/\mathbf{Z}}^1(\log \mathcal{Y}_S^{\text{red}}/\log S)$ and $\omega_{\mathcal{Y}/\mathbf{Z}}(\mathcal{Y}_S^{\text{red}} - \mathcal{Y}_S)$.*

PROOF. Recall that $\{\mathcal{Y}_i\}_{i \in \mathcal{I}}$ denote the irreducible components of the disjoint union of the special fibres $\mathcal{Y}_S^{\text{red}}$. From Proposition 3.1 in [CPT2] we know that the natural morphism

$$\omega : \Omega_{\mathcal{Y}/\mathbf{Z}} \rightarrow \Omega_{\mathcal{Y}/\mathbf{Z}}^1(\log \mathcal{Y}_S^{\text{red}}/\log S)$$

has the same kernel and cokernel as the natural map

$$a : \oplus_{p \in S} \mathcal{O}_{\mathcal{Y}}/p\mathcal{O}_{\mathcal{Y}} \rightarrow \oplus_{i \in \mathcal{I}} \mathcal{O}_{\mathcal{Y}_i}.$$

The result then follows on taking determinants, since $\omega_{\mathcal{Y}/\mathbf{Z}} \cong \det \Omega_{\mathcal{Y}/\mathbf{Z}}$. \square

PROOF OF THEOREM 7.8. For brevity we regard the isomorphic degree map γ of 7.7 as an identification and we again put $\mathcal{E} = \Omega_{\mathcal{Y}/\mathbf{Z}}^1(\log \mathcal{Y}_S^{\text{red}}/\log S)$.

For $0 \leq n \leq d$, the Duality Theorem in Section 11 of [H] gives a quasi-isomorphism of complexes

$$\text{R}\Gamma(\text{Hom}_{\mathcal{O}_{\mathcal{Y}}}(\wedge^n \mathcal{E}, \omega_{\mathcal{Y}/\mathbf{Z}}[d])) \cong \text{Hom}_{\mathbf{Z}}(\text{R}\Gamma(\wedge^n \mathcal{E}), \mathbf{Z})$$

and by standard Hodge theory we know that the induced isomorphisms on complex cohomology are isometries when the complex cohomology groups are endowed with their L^2 -metrics. Thus we see that

$$\begin{aligned}\chi_{L^2}(\mathrm{R}\Gamma(\mathrm{Hom}_{\mathcal{O}_Y}(\wedge^n \mathcal{E}, \omega_{Y/\mathbf{Z}}[d]))) &= \chi_{L^2}(\mathrm{Hom}_{\mathbf{Z}}(\mathrm{R}\Gamma \wedge^n \mathcal{E}, \mathbf{Z})) \\ &= \chi_{L^2}(\mathrm{R}\Gamma \wedge^n \mathcal{E})^{-1}.\end{aligned}\tag{27}$$

where for brevity we write $\chi_{L^2}(\mathrm{R}\Gamma \wedge^n \mathcal{E})$ in place of $\chi(\mathrm{R}\Gamma \wedge^n \mathcal{E}, \|\cdot\|_{L^2})$.

Next we observe that by Lemma 7.10, we know that $\mathrm{Hom}_{\mathcal{O}_Y}(\wedge^n \mathcal{E}, \omega_{Y/\mathbf{Z}}(\mathcal{Y}_p^{\mathrm{red}} - \mathcal{Y}_p)) \cong \wedge^{d-n} \mathcal{E}$. Thus we obtain a quasi-isomorphism

$$\mathrm{R}\Gamma(\wedge^{d-n} \mathcal{E}) \cong \mathrm{R}\Gamma(\mathrm{Hom}_{\mathcal{O}_Y}(\wedge^n \mathcal{E}, \omega_{Y/\mathbf{Z}}(\mathcal{Y}_S^{\mathrm{red}} - \mathcal{Y}_S)))$$

and, again by standard Hodge theory, we know that the induced isomorphisms on complex cohomology are isometries with respect to their L^2 -metrics. Thus we can write the number $\chi_{L^2}(\mathrm{R}\Gamma(\wedge^\bullet \mathcal{E}))^2$ as:

$$\prod_{n=0}^d \left[\chi_{L^2}(\mathrm{R}\Gamma(\wedge^n \mathcal{E}))^{(-1)^n} \cdot \chi_{L^2}(\mathrm{R}\Gamma(\mathrm{Hom}_{\mathcal{O}_Y}(\wedge^n \mathcal{E}, \omega_{Y/\mathbf{Z}}(\mathcal{Y}_S^{\mathrm{red}} - \mathcal{Y}_S)))^{(-1)^{d-n}} \right].$$

But this latter product can be rewritten as $\Pi_1 \cdot \Pi_2$ where Π_1 resp. Π_2 is the first resp. second of the following expressions:

$$\begin{aligned}\prod_{n=0}^d &\left[\chi_{L^2}(\mathrm{R}\Gamma(\wedge^n \mathcal{E}))^{(-1)^n} \cdot \chi_{L^2}(\mathrm{R}\Gamma(\mathrm{Hom}_{\mathcal{O}_Y}(\wedge^n \mathcal{E}, \omega_{Y/\mathbf{Z}}[d])))^{(-1)^n} \right] \\ \prod_{n=0}^d &\left[\chi_{L^2}(\mathrm{R}\Gamma(\mathrm{Hom}_{\mathcal{O}_Y}(\wedge^n \mathcal{E}, \omega_{Y/\mathbf{Z}})))^{-1} \cdot \chi_{L^2}(\mathrm{R}\Gamma(\mathrm{Hom}_{\mathcal{O}_Y}(\wedge^n \mathcal{E}, \omega_{Y/\mathbf{Z}}(\mathcal{Y}_S^{\mathrm{red}} - \mathcal{Y}_S)))) \right]^{(-1)^{d+n}}\end{aligned}$$

and we note that (27) implies that $\Pi_1 = 1$. Hence we may conclude that $\chi_{L^2}(\mathrm{R}\Gamma \wedge^\bullet \mathcal{E})^2$ is equal to Π_2 . In order to evaluate Π_2 we consider the exact sequences

$$0 \rightarrow \omega_{Y/\mathbf{Z}}(\mathcal{Y}_S^{\mathrm{red}} - \mathcal{Y}_S) \rightarrow \omega_{Y/\mathbf{Z}}(\mathcal{Y}_S^{\mathrm{red}}) \rightarrow \omega_{Y/\mathbf{Z}}(\mathcal{Y}_S^{\mathrm{red}})|_{\mathcal{Y}_S} \rightarrow 0$$

$$0 \rightarrow \omega_{Y/\mathbf{Z}} \rightarrow \omega_{Y/\mathbf{Z}}(\mathcal{Y}_S^{\mathrm{red}}) \rightarrow \omega_{Y/\mathbf{Z}}(\mathcal{Y}_S^{\mathrm{red}})|_{\mathcal{Y}_S^{\mathrm{red}}} \rightarrow 0$$

and we apply the exact functor $\mathrm{Hom}_{\mathcal{O}_Y}(\wedge^n \mathcal{E}, -)$ to get exact sequences

$$\begin{aligned}0 \rightarrow \mathrm{Hom}_{\mathcal{O}_Y}(\wedge^n \mathcal{E}, \omega_{Y/\mathbf{Z}}(\mathcal{Y}_S^{\mathrm{red}} - \mathcal{Y}_S)) &\rightarrow \mathrm{Hom}_{\mathcal{O}_Y}(\wedge^n \mathcal{E}, \omega_{Y/\mathbf{Z}}(\mathcal{Y}_S^{\mathrm{red}})) \\ &\rightarrow \mathrm{Hom}_{\mathcal{O}_Y}(\wedge^n \mathcal{E}, \omega_{Y/\mathbf{Z}}(\mathcal{Y}_S^{\mathrm{red}})|_{\mathcal{Y}_S}) \rightarrow 0\end{aligned}$$

and

$$\begin{aligned} 0 \rightarrow \mathrm{Hom}_{\mathcal{O}_{\mathcal{Y}}}(\wedge^n \mathcal{E}, \omega_{\mathcal{Y}/\mathbf{Z}}) &\rightarrow \mathrm{Hom}_{\mathcal{O}_{\mathcal{Y}}}(\wedge^n \mathcal{E}, \omega_{\mathcal{Y}/\mathbf{Z}}(\mathcal{Y}_S^{\mathrm{red}})) \\ &\rightarrow \mathrm{Hom}_{\mathcal{O}_{\mathcal{Y}}}(\wedge^n \mathcal{E}, \omega_{\mathcal{Y}/\mathbf{Z}}(\mathcal{Y}_S^{\mathrm{red}})|_{\mathcal{Y}_S^{\mathrm{red}}}) \rightarrow 0. \end{aligned}$$

Recall that h denotes the structure map $h : \mathcal{Y} \rightarrow \mathrm{Spec}(\mathbf{Z})$, m_i denotes the multiplicity of the component \mathcal{Y}_i in \mathcal{Y}_S and, as previously, for each i we let p_i denote the prime which supports \mathcal{Y}_i . For brevity we shall write $\mathrm{Hom}_{\mathcal{O}_{\mathcal{Y}}}(\wedge^\bullet \mathcal{E}, \omega_{\mathcal{Y}/\mathbf{Z}})$ for $\sum_n (-1)^n \mathrm{Hom}_{\mathcal{O}_{\mathcal{Y}}}(\wedge^n \mathcal{E}, \omega_{\mathcal{Y}/\mathbf{Z}})$ etc. It then follows from the above and from 5.5 and 5.6 that Π_2 is equal to

$$\begin{aligned} \nu \circ h_{S*} \left(\mathrm{Hom}_{\mathcal{O}_{\mathcal{Y}}}(\wedge^\bullet \mathcal{E}, \omega_{\mathcal{Y}/\mathbf{Z}}(\mathcal{Y}_S^{\mathrm{red}})|_{\mathcal{Y}_S^{\mathrm{red}}}) - \mathrm{Hom}_{\mathcal{O}_{\mathcal{Y}}}(\wedge^\bullet \mathcal{E}, \omega_{\mathcal{Y}/\mathbf{Z}}(\mathcal{Y}_S^{\mathrm{red}})|_{\mathcal{Y}_S}) \right) \\ = \prod_{i \in \mathcal{I}} p_i^{-(m_i-1)(-1)^d (c^d(\mathcal{E}) \cdot \mathcal{Y}_i)}. \end{aligned}$$

However, from 3.7 in [CPT2] (or see 5.1 in [CEPT2]), we know that

$$(-1)^d c^d(\mathcal{E}) \cdot \mathcal{Y}_i = \chi_c(\mathcal{Y}_i^*)$$

and so we have now shown

$$\chi_{L^2}(\mathrm{R}\Gamma \wedge^\bullet \mathcal{E})^2 = \prod_{i \in \mathcal{I}} p_i^{-(m_i-1)\chi_c(\mathcal{Y}_i^*)}.$$

Finally we need to allow for the fact that in the above we have used the L^2 -metric instead of the given Kähler metric. From the very definition of the Quillen metric, we know that

$$\log \chi(\mathrm{R}\Gamma \wedge^n \mathcal{E}, \wedge^n j^D) = \log \chi_{L^2}(\mathrm{R}\Gamma \wedge^n \mathcal{E}) + \tau(\wedge^n \Omega_Y, \wedge^n j^D)$$

where $\tau(\wedge^n \Omega_Y, \wedge^n j^D)$ denotes the analytic torsion associated to $\wedge^n \Omega_Y$ with respect to the metric $\wedge^n j^D$. But Theorem 3.1 in [RS] shows that

$$\sum_n (-1)^n \tau(\wedge^n \Omega_Y, \wedge^n j^D) = 0 \tag{28}$$

and so we have now shown

$$\chi(\mathrm{R}\Gamma \wedge^\bullet \mathcal{E}, \wedge^\bullet j^D) = \prod_{i \in \mathcal{I}} p_i^{-(m_i-1)\chi_c(\mathcal{Y}_i^*)}.$$

This then completes the proof of Theorem 7.8. \square

Observe that Theorems 7.8 and 7.9 show that

$$\gamma \circ \chi \left(\mathrm{R}\Gamma(\wedge^\bullet \Omega_{\mathcal{Y}/\mathbf{Z}}^1(\log \mathcal{Y}_S^{\mathrm{red}}/\log S), \wedge^\bullet j_Q^D) \right)^2 = \xi_S(2) \varepsilon_{0,S}(\mathcal{Y}, 2)^{-1}.$$

We conclude this section by showing that the right hand factor $\xi_S(2)$ in the above can be removed by twisting the sheaf $\Omega_{\mathcal{Y}/\mathbf{Z}}^1(\log \mathcal{Y}_S^{\text{red}}/\log S)$ by $\mathcal{O}_{\mathcal{Y}}(-\mathcal{Y}_S)$.

Theorem 7.11 $\gamma \circ \chi \left(R\Gamma(\wedge^\bullet \Omega_{\mathcal{Y}/\mathbf{Z}}^1(\log \mathcal{Y}_S^{\text{red}}/\log S)(-\mathcal{Y}_S)), \wedge^\bullet j_Q^D \right)^2 = \varepsilon_{0,S}(\mathcal{Y}, 2)^{-1}.$

PROOF. Since for each $i \geq 0$

$$\wedge^i(\Omega_{\mathcal{Y}/\mathbf{Z}}^1(\log \mathcal{Y}_S^{\text{red}}/\log S)(-\mathcal{Y}_S)) = \wedge^i \Omega_{\mathcal{Y}/\mathbf{Z}}^1(\log \mathcal{Y}_S^{\text{red}}/\log S) \otimes \mathcal{O}_{\mathcal{Y}}(-i\mathcal{Y}_S)$$

we obtain an exact sequence of complexes of sheaves

$$0 \rightarrow \wedge^\bullet(\Omega_{\mathcal{Y}/\mathbf{Z}}^1(\log \mathcal{Y}_S^{\text{red}}/\log S)(-\mathcal{Y}_S)) \rightarrow \wedge^\bullet \Omega_{\mathcal{Y}/\mathbf{Z}}^1(\log \mathcal{Y}_S^{\text{red}}/\log S) \rightarrow \mathcal{G}^\bullet \rightarrow 0$$

where for $0 \leq i \leq d$

$$\mathcal{G}^i = \wedge^i \Omega_{\mathcal{Y}/\mathbf{Z}}^1(\log \mathcal{Y}_S^{\text{red}}/\log S)|_{i\mathcal{Y}_S}$$

and so by (5.5), (5.6) and the equality displayed prior to (7.11)

$$\gamma \circ \chi \left(R\Gamma(\wedge^\bullet \Omega_{\mathcal{Y}/\mathbf{Z}}^1(\log \mathcal{Y}_S^{\text{red}}/\log S)(-\mathcal{Y}_S)), \wedge^\bullet j_Q^D \right)^2 = \xi_S(2) \cdot \varepsilon_{0,S}(\mathcal{Y}, 2)^{-1} \cdot \chi(\nu(\mathcal{G}^\bullet))^2$$

and for $0 \leq i \leq d$

$$\chi(\nu(\mathcal{G}^i))^2 = \prod_{p \in S} p^{2f_*(\mathcal{G}^i)} = \prod_{p \in S} p^{2i\chi(\Omega_{\mathcal{Y}_{\mathbf{Q}}}^i)}$$

since $\Omega_{\mathcal{Y}/\mathbf{Z}}^1(\log \mathcal{Y}_S^{\text{red}}/\log S)_{\mathbf{Q}} = \Omega_{\mathcal{Y}_{\mathbf{Q}}}$. However, by Serre duality we know that $(-1)^{d-i} \chi(\Omega_{\mathcal{Y}_{\mathbf{Q}}}^{d-i}) = (-1)^i \chi(\Omega_{\mathcal{Y}_{\mathbf{Q}}}^i)$ and so we see that

$$\sum_{i=0}^d (-1)^i i \cdot \chi(\Omega_{\mathcal{Y}_{\mathbf{Q}}}^i) = \sum_{i=0}^d (-1)^{d-i} i \cdot \chi(\Omega_{\mathcal{Y}_{\mathbf{Q}}}^{d-i}) = \sum_{i=0}^d (-1)^i (d-i) \cdot \chi(\Omega_{\mathcal{Y}_{\mathbf{Q}}}^i)$$

hence

$$\sum_{i=0}^d (-1)^i 2i \cdot \chi(\Omega_{\mathcal{Y}_{\mathbf{Q}}}^i) = d \cdot \chi(\mathcal{Y}_{\mathbf{Q}})$$

which therefore shows that

$$\prod_i \chi(\nu(\mathcal{G}^i))^{2(-1)^i} = \prod_{p \in S} p^{2i(-1)^i \chi(\Omega_{\mathcal{Y}_{\mathbf{Q}}}^i)} = \prod_{p \in S} p^{d\chi(\mathcal{Y}_{\mathbf{Q}})} = \xi_S(2)^{-1}$$

as required. \square

8 Differentials

In this the final section of the article we construct arithmetic classes associated to the sheaf of (regular) differentials $\Omega_{\mathcal{X}/\mathbf{Z}}^1$. Since $\Omega_{\mathcal{X}/\mathbf{Z}}^1$ is not in general locally free over $O_{\mathcal{X}}$, we resolve it by locally free G - \mathcal{X} sheaves as follows: we choose a G -equivariant embedding $i : \mathcal{X} \rightarrow \mathcal{P}$ of \mathcal{X} into a projective bundle \mathcal{P} over $\mathrm{Spec}(\mathbf{Z})$. The sheaf of differentials $\Omega_{\mathcal{X}/\mathbf{Z}}^1$ then has a resolution by locally free G - \mathcal{X} sheaves

$$0 \rightarrow N^* \rightarrow P \xrightarrow{\pi} \Omega_{\mathcal{X}/\mathbf{Z}}^1 \rightarrow 0 \quad (29)$$

where $P = i^* \Omega_{\mathcal{P}/\mathbf{Z}}^1$, and where N^* denotes the conormal bundle associated to the regular embedding i . Let \mathcal{F}^\bullet denote the length two complex

$$\mathcal{F}^\bullet : N^* \rightarrow P$$

where the term P is deemed to have degree zero. Thus we may view π as inducing a quasi-isomorphism of complexes, which we abusively also denote π ,

$$\pi : \mathcal{F}^\bullet \rightarrow \Omega_{\mathcal{X}/\mathbf{Z}}^1.$$

Here we further abuse notation and write $\Omega_{\mathcal{X}/\mathbf{Z}}^1$ for the complex which is $\Omega_{\mathcal{X}/\mathbf{Z}}^1$ in degree zero and which is zero elsewhere.

For $j \geq 0$, recall that we have the Dold-Puppe exterior power functors \bigwedge^j defined on bounded complexes of locally free G - \mathcal{X} sheaves and which take quasi-isomorphisms to quasi-isomorphisms. (See [CPT3] for an account of these functors which is particularly well-suited to their use in this paper.)

We then endow the equivariant determinant of cohomology of the complex $\bigwedge^j(\mathcal{F}^\bullet)$ with the metrics $\phi_{j\bullet}$ induced, via $\bigwedge^j(\pi_{\mathbf{C}})$, from the $\bigwedge^j h_{Q\bullet}^D$ on the determinants of cohomology of $\Omega_{X/\mathbf{C}}^j$; we then define arithmetic classes

$$\chi \left(\mathrm{R}\Gamma \lambda^j \Omega_{\mathcal{X}/\mathbf{Z}}^1, \bigwedge^j h_Q^D \right) := \chi \left(\mathrm{R}\Gamma \bigwedge^j (\mathcal{F}^\bullet), \phi_j \right) \quad (30)$$

$$\chi \left(\mathrm{R}\Gamma \lambda^\bullet \Omega_{\mathcal{X}/\mathbf{Z}}^1, \bigwedge^\bullet h_Q^D \right) := \prod_{j=0}^d \chi \left(\mathrm{R}\Gamma \lambda^j \Omega_{\mathcal{X}/\mathbf{Z}}, \bigwedge^j h_Q^D \right)^{(-1)^j}. \quad (31)$$

Note that here the use of the symbols λ^j is entirely symbolic; however, it is important to observe that the lefthand classes are independent of the chosen embedding $i : \mathcal{X} \rightarrow \mathcal{P}$: indeed, for a further embedding i' , with the obvious notation, $\bigwedge^j(\mathcal{F}'^\bullet)$ is quasi-isomorphic

to $\wedge^j(\mathcal{F}^\bullet)$; furthermore their metrics on the determinant of cohomology match under the corresponding quasi-isomorphism; hence by 3.9 the arithmetic classes coincide.

The equivariant Arakelov-Euler characteristic $\chi(\mathrm{R}\Gamma \lambda^\bullet \Omega_{\mathcal{X}/\mathbf{Z}}^1, \wedge^\bullet h_Q^D)$ is the principal object of study in this section. Our aim here is to relate it to the epsilon constant $\varepsilon(\mathcal{Y})$, whose definition we now briefly recall. Let $\mathbf{A}_\mathbf{Q}$ denote the ring of rational adeles; let $\psi = \prod_v \psi_v$ denote a non-trivial additive character of $\mathbf{A}_\mathbf{Q}/\mathbf{Q}$; let dx denote the Haar measure on $\mathbf{A}_\mathbf{Q}/\mathbf{Q}$ such that $\int_{\mathbf{A}_\mathbf{Q}/\mathbf{Q}} dx = 1$ and let $dx = \prod_v dx_v$ be a factorisation of dx into local Haar measures dx_v with the property that $\int_{\mathbf{Z}_v} dx_v = 1$ for almost all v . Recall from 3.1.1 in [CEPT1] that for $\theta \in R_G$,

$$\varepsilon_v(\mathcal{Y}, \theta, \psi_v, dx_v, l_v) = \varepsilon_{0,v}(\mathcal{Y}, \theta, \psi_v, dx_v, l_v) \varepsilon(\mathcal{Y}_v, \theta).$$

Here if $v < \infty$ then $\varepsilon(\mathcal{Y}_v, \theta)$ is the epsilon constant associated to the special fibre \mathcal{Y}_v and if $v = \infty$ then we take $\varepsilon(\mathcal{Y}_v, \theta) = 1$. We then set

$$\varepsilon(\mathcal{Y}, \theta) = \prod_v \varepsilon_v(\mathcal{Y}_v, \theta, \psi_v, dx_v, l_v).$$

Note that in this product almost all terms are 1 and moreover this product is independent of choices of additive character and Haar measure. Thus in the lefthand term we shall abuse notation and henceforth we shall not overtly mention the choices of auxiliary primes l_v .

For future reference we now need to gather together some standard results on fibral epsilon constants.

For this we require a minor variant on the notation introduced prior to Theorem 5.10. As previously, given a prime number p we fix a field embedding $h : \overline{\mathbf{Q}} \rightarrow \overline{\mathbf{Q}}_p$, we put $(\overline{\mathbf{Q}})_p = \overline{\mathbf{Q}} \otimes \overline{\mathbf{Q}}_p$ and we let $J_f \rightarrow (\overline{\mathbf{Q}})_p^\times$ denote the map given by projection to the p th coordinate; given $x \in J_f$, we shall write x_p for the p -component of x in $(\overline{\mathbf{Q}})_p^\times$. Let $|-|_p : \overline{\mathbf{Q}}_p^\times \rightarrow p^\mathbf{Q}$ denote the p -adic absolute value which is normalised so that $|p|_p = p^{-1}$. We shall use the terminology of Definition 5.6 in [C] and for $f \in \mathrm{Hom}_\Omega(R_G, (\overline{\mathbf{Q}})_p^\times)$ we say that $|f|_p$ is *well-defined* if $|h^*(f)|_p$ takes values in $p^\mathbf{Z}$; in this case it follows that $|h^*(f)|_p$ respects Ω_p -action and we then write

$$|f|_p = h^{*-1} |h^* f|_p.$$

Theorem 8.1 *For each prime number p , $|\varepsilon(\mathcal{Y}_p)|_p$ and $|\varepsilon(b_p)|_p$ are well-defined. Writing U_p for the open sub-scheme $Y_p - b_p$, we have $\varepsilon(\mathcal{Y}_p) = \varepsilon(U_p)\varepsilon(b_p)$ and*

$$\varepsilon(U_p)_p |\varepsilon(U_p)|_p \in \mathrm{Det}(\mathbf{Z}_p[G]^\times)$$

and for a prime number $q \neq p$

$$\varepsilon(U_p)_q \in \text{Det}(\mathbf{Z}_q[G]^\times).$$

PROOF. See [C] 5.7, 5.13 and 5.12. \square

In order to make precise the fundamental relationship between $\chi(\text{R}\Gamma\lambda^\bullet\Omega_{\mathcal{X}/\mathbf{Z}}^1, \wedge^\bullet h_{Q\bullet}^D)$ and $\varepsilon(\mathcal{Y})$, we now need to introduce the *arithmetic ramification class*, which may be viewed as an arithmetic counterpart of the ramification class occurring in Theorem 1.1 in [CPT1].

Definition 8.2 Let $\text{AR}(\mathcal{X}) \in A(\mathbf{Z}[G])$ be the arithmetic class which is represented by the idele valued character function β , given by the rule that β has trivial archimedean coordinate and at a finite prime q

$$\beta_q = \varepsilon(b) | \varepsilon(b_q) |_q$$

where b_q denotes the union of the components of b which are supported by q .

We are now in a position to be able to state the main result of this article:

Theorem 8.3 Let \mathfrak{d} be the arithmetic class $\chi(\text{R}\Gamma\lambda^\bullet\Omega_{\mathcal{X}/\mathbf{Z}}^1, \wedge^\bullet h_Q^D)$. Then $\mathfrak{d}^s \cdot \text{AR}^s(\mathcal{X})^{-1}$ is a rational class and

$$\theta(\mathfrak{d}^s \cdot \text{AR}^s(\mathcal{X})^{-1}) = \varepsilon^s(\mathcal{Y})^{-1}.$$

As a first step towards the proof of this theorem, we use results from [CPT2] to show that it will suffice to establish the corresponding result after passage to degree zero:

Theorem 8.4 The class $\widetilde{\mathfrak{d}}^s \cdot \widetilde{\text{AR}}^s(\mathcal{X})$ is a rational class and

$$\theta(\widetilde{\mathfrak{d}}^s \cdot \widetilde{\text{AR}}^s(\mathcal{X})^{-1}) = \widetilde{\varepsilon}^s(\mathcal{Y})^{-1}.$$

We begin by showing that Theorem 8.4 implies Theorem 8.3; we then conclude the article by establishing Theorem 8.4.

Suppose then that Theorem 8.4 holds. From 4.E, with the notation of 4.2, we know that

$$\mathfrak{d} = \widetilde{\mathfrak{d}} \cdot \text{Ind}(\mathfrak{d}_0).$$

By 5.7 in [C] we know that $\varepsilon(b_p, 1_G)$ is \pm an integral power of p ; hence we see that β and $\widetilde{\beta}$ represent the same class in $A(\mathbf{Z}[G])$ and so $\widetilde{\text{AR}}^s(\mathcal{X}) = \text{AR}^s(\mathcal{X})$.

Theorem 8.5 \mathfrak{d}_0^s is a rational class and $\theta(\mathfrak{d}_0^s) = \varepsilon(\mathcal{Y}, 2 \cdot 1_G)^{-1}$.

Proof. We endow P and N^* in (29) with G -invariant hermitian metrics and denote the resulting hermitian bundles by \widehat{P} and \widehat{N}^* . We then let η_1 denote the Bott-Chern class associated to the exact sequence (29), where Ω_Y^1 is endowed with the hermitian metric h^D , and we put $\widehat{\Omega} = \widehat{P} - \widehat{N}^* + \eta_1$ in the arithmetic Grothendieck group $\widehat{K}_0(\mathcal{Y})$ (see for instance II, Sect. 6 in [GS1]); we recall *loc. cit.* that $\widehat{K}_0(\mathcal{Y})$ has a natural structure of a λ -ring and we write \widehat{f}_* for the push forward map from $\widehat{K}_0(\mathcal{Y})$ to $\widehat{K}_0(\text{Spec}(\mathbf{Z}))$. Because \mathcal{Y} is regular, we know that $\widehat{K}_0(\mathcal{Y})$ is naturally isomorphic to $\widehat{K}'_0(\mathcal{Y})$ the Grothendieck group of coherent hermitian sheaves (see Lemma 13 in [GS2]). Thus we also have a natural map from the Grothendieck group of torsion \mathcal{Y} -sheaves supported on S , denoted $\widehat{K}_0^S(\mathcal{Y})$, to $\widehat{K}_0(\mathcal{Y})$. Recall that $\widehat{K}_0^S(\mathcal{Y})$ is a module over the Grothendieck group of locally free \mathcal{Y} -sheaves $\widehat{K}_0(\mathcal{Y})$.

In Theorem 1.3 of [CPT1], with slightly different notation, it is shown that

$$\sum_{i=0}^d \gamma \circ \chi \circ \widehat{f}_*((-1)^i \lambda^i \widehat{\Omega}) = |\varepsilon(\mathcal{Y}, 1_G)|^{-1}$$

whereas the class that we now wish to study is

$$\gamma \circ \chi(\text{R}\Gamma \lambda^\bullet \Omega_{\mathcal{Y}/\mathbf{Z}}^1, \wedge^\bullet h_{Q,1}^D) = \sum_{i=0}^d \gamma \circ \chi \circ \widehat{f}_*((-1)^i (\lambda^i (\widehat{P} - \widehat{N}^*) + \eta_2^{(i)}))$$

where the i th exterior power $\wedge^i \Omega_Y = \Omega_Y^i$ carries the hermitian metric $\wedge^i h^D$, the terms of $\wedge^i \mathcal{F}_C^\bullet$ carry the metrics coming from \widehat{P} and \widehat{N}^* , and where $\eta_2^{(i)}$ is the Bott-Chern class associated to the exact sequence of hermitian bundles $\wedge^i \widehat{\mathcal{F}}_C^\bullet \rightarrow \wedge^i \widehat{\Omega}_Y$. As our first step in proving the theorem, we will show that in $\widehat{K}_0(\mathcal{Y})$

$$\lambda^i(\widehat{P} - \widehat{N}^*) + \eta_2^{(i)} = \lambda^i(\widehat{P} - \widehat{N}^* + \eta_1) \quad (32)$$

which will then imply that

$$\gamma \circ \chi(\text{R}\Gamma \lambda^\bullet \Omega_{\mathcal{Y}/\mathbf{Z}}^1, \wedge^\bullet h_{Q,1}^D) = |\varepsilon(\mathcal{Y}, 1_G)|^{-1} \quad (33)$$

From Lemma 7.10 we know that there is an exact sequence

$$0 \rightarrow K \oplus N^* \rightarrow P \rightarrow \Omega_{\mathcal{Y}/\mathbf{Z}}^1(\log \mathcal{Y}_S^{\text{red}} / \log S) \rightarrow C \rightarrow 0$$

where K and C explicitly determined torsion $\mathcal{O}_{\mathcal{Y}}$ -modules supported on S . Thus for each $i, 0 \leq i \leq d$, we have an equality in $\widehat{K}'_0(\mathcal{Y})$

$$\lambda^i(\widehat{P} - \widehat{N}^* - \widehat{K} + \eta_1) = \lambda^i \left(\widehat{\Omega}_{\mathcal{Y}/\mathbf{Z}}^1(\log \mathcal{Y}_S^{\text{red}} / \log S) - \widehat{C} \right)$$

which we can rewrite as

$$\lambda^i(\widehat{\Omega}) + T_1^{(i)} = \lambda^i \left(\widehat{\Omega}_{\mathcal{Y}/\mathbf{Z}}^1(\log \mathcal{Y}_S^{\text{red}} / \log S) \right) + T_2^{(i)}$$

where $T_1^{(i)}$ and $T_2^{(i)}$ are the following torsion classes

$$T_1^{(i)} = \sum_{a+b=i, b>0} \lambda^a(P - N^*) \lambda^b(-K),$$

$$T_2^{(i)} = \sum_{a+b=i, b>0} \lambda^a \left(\widehat{\Omega}_{\mathcal{Y}/\mathbf{Z}}^1(\log \mathcal{Y}_S^{\text{red}} / \log S) \right) \lambda^b(-C).$$

Next we consider the quasi-isomorphism of the Dold-Puppe exterior powers (where P and $\Omega_{\mathcal{Y}/\mathbf{Z}}^1(\log \mathcal{Y}_S^{\text{red}} / \log S)$ are both deemed to have degree zero)

$$\wedge^i(K^\bullet \oplus N^* \rightarrow P) \cong \bigwedge^i \left(\Omega_{\mathcal{Y}/\mathbf{Z}}^1(\log \mathcal{Y}_S^{\text{red}} / \log S) \rightarrow C^\bullet \right)$$

and where K^\bullet and C^\bullet denote locally free resolutions of K and C . Hence filtering the complex $\wedge^i(K^\bullet \oplus N^* \rightarrow P)$, by terms $\wedge^a(N^* \rightarrow P) \otimes \wedge^{i-a}(K^\bullet[1])$, and filtering the complex $\wedge^i(\Omega_{\mathcal{Y}/\mathbf{Z}}^1(\log \mathcal{Y}_S^{\text{red}} / \log S) \rightarrow C^\bullet)$ by terms $\wedge^a(\Omega_{\mathcal{Y}/\mathbf{Z}}^1(\log \mathcal{Y}_S^{\text{red}} / \log S)) \otimes \wedge^{i-a}(C^\bullet[-1])$ (see p. 26 in [S]), we obtain an equality in $\widehat{K}'_0(\mathcal{Y})$

$$T_1^{(i)} + \lambda^i(\widehat{P} - \widehat{N}^*) + \eta_2^{(i)} = \lambda^i \left(\widehat{\Omega}_{\mathcal{Y}/\mathbf{Z}}^1(\log \mathcal{Y}_S^{\text{red}} / \log S) \right) + T_2^{(i)}$$

which now establishes (32).

Now each $\varepsilon(\mathcal{Y}, 1_G)$ is a rational number and so by Theorem 7.9, $\varepsilon(\mathcal{Y}, 2 \cdot 1_G) \in \mathbf{Q}_{>0}$. Thus by Lemma 7.7 we see that if we can show

$$\mathfrak{d}_0 = \chi(\mathrm{R}\Gamma \lambda^\bullet \Omega_{\mathcal{Y}/\mathbf{Z}}^1, \wedge^\bullet h_{Q,1}^D) \quad (34)$$

then it will follow that \mathfrak{d}_0 is a rational class and that

$$\theta(\mathfrak{d}_0^s) = \varepsilon(\mathcal{Y}, 2 \cdot 1_G)^{-1}.$$

With the notation of 7.6 above and by the very definition of \mathfrak{d}_0 (see 4.2),

$$\mathfrak{d}_0 = \chi(\mathrm{R}\Gamma \lambda^\bullet(\mathcal{F}^\bullet)^G, \wedge^\bullet h_{Q,1}^D).$$

Thus we are now required to show that for each $j, 0 \leq j \leq d$, the natural map $\wedge^j(\mathcal{F}^\bullet)^G$ to $(\wedge^j \mathcal{F}^\bullet)^G$ is a quasi-isomorphism. To see this it will suffice to show the result after passing to a flat neighbourhood of each closed point of y of \mathcal{Y} . Writing $\mathcal{X}' \rightarrow \mathcal{Y}'$ for the resulting

base change to such a neighbourhood, we let x' resp. y' denote a closed point of \mathcal{X}' resp. \mathcal{Y}' above y . From Theorem A.1 and Lemma A.2 in [CEPT1], we know that, for a suitable choice of neighbourhood, \mathcal{X}' contains $(G : I_{x'})$ disjoint irreducible components which are permuted transitively by G and where the component which contains x' has stabiliser $I_{x'}$, the inertia group of x' . If B_1, \dots, B_q are the distinct irreducible components of the inverse image $\pi^{-1}(b)$ which contain the image of x' on \mathcal{X} , then

$$I_{x'} = I_1 \oplus \dots \oplus I_q$$

where I_i denotes the inertia group of the generic point of B_i ; moreover, each I_i carries a faithful abelian character ϕ_i given by the action of I_i on the cotangent space of the generic point of B_i . To be somewhat more precise, there are integers n_1, \dots, n_{d+1} coprime to the residual characteristic of y so that, after base extension by a suitable affine flat neighbourhood $\text{Spec}(R)$, the connected open neighbourhood V of $\mathcal{X}' = \mathcal{X} \times \text{Spec}(T)$ containing x' is the spectrum of

$$\frac{R[U_1, \dots, U_{d+1}]}{(U_1^{n_1} - a_1, \dots, U_1^{n_{d+1}} - a_{d+1})}.$$

Here a_1, \dots, a_{d+1} form a system of regular parameters of \mathcal{Y}' ; moreover there are integers m_i for $1 \leq i \leq d+1$ with each m_i coprime to the residual characteristic, p say, of y , and with the property that $a_1^{m_1} \dots a_{d+1}^{m_{d+1}} = p$. Here, after reordering if necessary, the characters ϕ_i are given by the action of I_i on U_i . It now follows that $\Omega_{V/R}^1$ sits in an exact sequence

$$0 \rightarrow K^\bullet \rightarrow \Omega_{V/R}^1 \rightarrow 0$$

with

$$K^\bullet : \mathcal{O}_V dr \rightarrow \bigoplus_{i=1}^{d+1} \mathcal{O}_V dU_i$$

and where $r = a_1^{m_1} \dots a_{d+1}^{m_{d+1}} - p$. In the sequel for brevity we shall write $K^\bullet = L \rightarrow E$. Since the restriction $\mathcal{F}^\bullet|_V$ is quasi-isomorphic to K^\bullet we are now reduced to showing that $\wedge^m(K^\bullet)^I \simeq (\wedge^m K^\bullet)^I$ for all $m \geq 0$ and for $I = I_{x'}$. This now follows easily since we know (see for instance Sect. 3 in [CPT3]) that the complex $\wedge^m K^\bullet$ is constituted entirely of terms which are tensor products of modules of the form $\wedge^n L$ times either one or no terms of the form $\wedge^n E$; the result then follows because $L \cong \mathcal{O}_V$, as I -modules, and because, for any non-negative n , $\wedge^n(E^I) \cong (\wedge^n E)^I$ (using the fact that the ϕ_i come from the distinct components in a direct sum decomposition). \square

PROOF OF THEOREM 8.4. We write $\mathcal{X}_S^{\text{red}} = \bigsqcup_{p \in S} \mathcal{X}_p^{\text{red}}$, let $i_S^{\text{red}} : \mathcal{X}_S^{\text{red}} \rightarrow \mathcal{X}$ denote the associated closed embedding and we let U_S denote the complement of $\mathcal{X}_S^{\text{red}}$ in \mathcal{X} .

Composing the quasi-isomorphism $\pi : \mathcal{F}^\bullet \simeq \Omega_{\mathcal{X}/\mathbf{Z}}^1$ with the natural homomorphism $\omega : \Omega_{\mathcal{X}/\mathbf{Z}}^1 \rightarrow \Omega_{\mathcal{X}/\mathbf{Z}}^1(\log \mathcal{X}_S^{\text{red}}/\log S)$, which is an isomorphism over U_S , we get a chain map

$$\pi' : \mathcal{F}^\bullet \rightarrow \Omega_{\mathcal{X}/\mathbf{Z}}^1(\log \mathcal{X}_S^{\text{red}}/\log S)$$

which is a surjective quasi-isomorphism over U_S . Hence for $i \geq 0$ we obtain maps

$$\wedge^i \pi' : \wedge^i \mathcal{F}^\bullet \rightarrow \wedge^i \Omega_{\mathcal{X}/\mathbf{Z}}^1(\log \mathcal{X}_S^{\text{red}}/\log S)$$

which are surjective quasi-isomorphisms over U_S . Let

$$\mathcal{A}_i^\bullet = \ker(\wedge^i \pi') \quad \text{and} \quad \mathcal{B}_i^\bullet = \text{coker}(\wedge^i \pi')$$

so that \mathcal{B}_i^\bullet and the cohomology sheaves $\mathcal{H}^j(\mathcal{A}_i^\bullet)$ of the complex \mathcal{A}_i^\bullet are all supported entirely over S . Let \mathfrak{I} denote the ideal sheaf of $\mathcal{O}_{\mathcal{X}}$ associated to the closed subscheme $\mathcal{X}_S^{\text{red}}$; we then write $[\mathcal{H}^j(\mathcal{A}_i^\bullet)]$ for the finite sum

$$\sum_{n \geq 0} (\mathfrak{I}^n \mathcal{H}^j(\mathcal{A}_i^\bullet) / \mathfrak{I}^{n+1} \mathcal{H}^j(\mathcal{A}_i^\bullet))$$

in $G_0(G, \mathcal{X}_S^{\text{red}})$ and put

$$[\mathcal{H}^\bullet(\mathcal{A}_i^\bullet)] = \sum_j (-1)^j [\mathcal{H}^j(\mathcal{A}_i^\bullet)] \in G_0(G, \mathcal{X}_S^{\text{red}}).$$

We endow the equivariant determinants of cohomology of \mathcal{A}_i^\bullet and \mathcal{B}_i^\bullet with the trivial metrics τ_\bullet . Then from 6.4 and 5.6 we know that

$$\begin{aligned} & \widetilde{\chi}(\text{R}\Gamma \wedge^i \mathcal{F}^\bullet, \wedge^i h^D) \cdot \widetilde{\chi}\left(\text{R}\Gamma \wedge^i \Omega_{\mathcal{X}/\mathbf{Z}}^1(\log \mathcal{X}_S^{\text{red}}/\log S), \wedge^i h^D\right)^{-1} \\ &= \widetilde{\chi}(\text{R}\Gamma \mathcal{A}_i^\bullet, \tau_\bullet) \cdot \widetilde{\chi}(\text{R}\Gamma \mathcal{B}_i^\bullet, \tau_\bullet)^{-1} \\ &= \widetilde{\nu \circ f_{S*}^{\text{red}}}([\mathcal{H}^\bullet(\mathcal{A}_i^\bullet)] - [\mathcal{H}^\bullet(\mathcal{B}_i^\bullet)]). \end{aligned} \tag{35}$$

Since the class $F = (-1)^d \sum_i (-1)^i ([\mathcal{H}^\bullet(\mathcal{A}_i^\bullet)] - [\mathcal{H}^\bullet(\mathcal{B}_i^\bullet)])$ in $G_0(G, \mathcal{X}_S^{\text{red}})$ has the property that its image in $G_0(G, \mathcal{X}) = K_0(G, \mathcal{X})$

$$\begin{aligned} i_{S*}^{\text{red}} F &= (-1)^d \sum_i (-1)^i \left([\wedge^i \mathcal{F}^\bullet] - \left[\wedge^i \Omega_{\mathcal{X}/\mathbf{Z}}^1(\log \mathcal{X}_S^{\text{red}}/\log S) \right] \right) \\ &= c^d(\Omega_{\mathcal{X}/\mathbf{Z}}) - c^d(\Omega_{\mathcal{X}/\mathbf{Z}}^1(\log \mathcal{X}_S^{\text{red}}/\log S)) \end{aligned} \tag{36}$$

we may take $F = \oplus_{p \in S} F_p$ to be the class F in 6(a) of [CPT1].

Definition 8.6 For each $i \in \mathcal{I}$ we set $\mathcal{X}_i = \pi^{-1}(\mathcal{Y}_i)$; thus \mathcal{X}_i is a smooth projective variety over \mathbf{F}_{p_i} of dimension d which carries a tame G -action. More generally for each non-empty subset \mathcal{J} of \mathcal{I} we define

$$\mathcal{Y}_{\mathcal{J}} = \cap_{j \in \mathcal{J}} \mathcal{Y}_j, \quad \mathcal{X}_{\mathcal{J}} = \cap_{j \in \mathcal{J}} \mathcal{X}_j$$

so that each $\mathcal{X}_{\mathcal{J}}$ is either empty or is a smooth projective variety of dimension $d + 1 - |\mathcal{J}|$. Again $\mathcal{X}_{\mathcal{J}}$ carries a tame G -action and the branch locus of the cover $\mathcal{X}_{\mathcal{J}}/\mathcal{Y}_{\mathcal{J}}$ is a divisor with strict normal crossings. Let \mathcal{I}_p denote the subset of those $i \in \mathcal{I}$ such that $p_i = p$. For $\mathcal{J} \subset \mathcal{I}_p$ we write $f_{\mathcal{J}}$ for the structure map $f_{\mathcal{J}} : \mathcal{X}_{\mathcal{J}} \rightarrow \text{Spec}(\mathbf{F}_p)$ and as per 6.b in [CPT1] we set

$$\begin{aligned} \Psi_p(\mathcal{X}_{\mathcal{J}}/\mathcal{Y}_{\mathcal{J}}) &= (-1)^{d-|\mathcal{J}|} f_{\mathcal{J}*}(c^{d-|\mathcal{J}|}(\Omega_{\mathcal{X}_{\mathcal{J}}/\mathbf{F}_p})) \quad \text{in } K_0(\mathbf{F}_p[G]) \\ \Psi_p &= \sum_{\phi \neq \mathcal{J} \subset \mathcal{I}} (-1)^{|\mathcal{J}|+1} \Psi_p(\mathcal{X}_{\mathcal{J}}/\mathcal{Y}_{\mathcal{J}}) \end{aligned}$$

and

$$\Psi = \oplus_{p \in S} \Psi_p \in \oplus_{p \in S} K_0(\mathbf{F}_p[G]).$$

Theorem 8.7 (a) The classes $f_{p*}F_p$ and $(-1)^d \Psi_p$ differ by the class of a free $\mathbf{F}_p[G]$ -module in $K_0(\mathbf{F}_p[G])$, and so

$$\tilde{\nu}(f_{p*}F_p) = \tilde{\nu}(\Psi_p)^{(-1)^d}.$$

(b) The class $\tilde{\nu}(\Psi_p)$ is represented by the idele valued character homomorphism δ_p

$$\delta(\theta)_v = \begin{cases} |\varepsilon_p(\mathcal{Y}_p, \bar{\theta})|_p & \text{if } v = p; \\ 1 & \text{if } v \neq p. \end{cases}$$

PROOF. This is the content of (A) and (B) in 6(b) of [CPT1], but note that, as explained in 4.B, we adopt the opposite convention on the representative of a class in $K_0T(\mathbf{Z}[G])$ to that used in [CPT1]. \square

We are now in a position to complete the proof of Theorem 8.4. From (35), (36) and part (a) of Theorem 8.7 we know that

$$\tilde{\mathfrak{d}} \cdot \tilde{\mathfrak{c}}^{-1} = \tilde{\nu}(i_{S*}(F))^{(-1)^d} = \tilde{\nu}(\Psi)$$

and by part (b) of Theorem 8.7 we know that $\tilde{\nu}(\Psi)$ is represented by the finite idele valued homomorphism on characters $\delta = \prod_p \delta_p$. From Theorem 7.5 we know that $\tilde{\mathfrak{c}}$ is a rational

class and that $\theta(\tilde{\mathfrak{r}}^s) = \tilde{\varepsilon}_0^s(\mathcal{Y})^{-1}$; it therefore follows that $\tilde{\mathfrak{d}}^s$ is represented by the character function with trivial Archimedean coordinate and whose finite coordinate is

$$\tilde{\varepsilon}_0^s(\mathcal{Y})^{-1} \left[\prod_{p \in S} |\tilde{\varepsilon}^s(\mathcal{Y}_p)|_p \tilde{\varepsilon}^s(\mathcal{Y}_p) \tilde{\varepsilon}^s(\mathcal{Y}_p)^{-1} \right].$$

Therefore, to complete the proof of Theorem 8.4, we are now reduced to showing:

Proposition 8.8 *The character function $\prod_{p \in S} |\varepsilon(\mathcal{Y}_p)|_p \varepsilon(\mathcal{Y}_p)$ represents the arithmetic ramification class $\text{AR}(\mathcal{X})$.*

PROOF. For $f, g \in \text{Hom}_\Omega(R_G, J_f)$ we write $f \sim g$ if f and g represent the same class in $A(\mathbf{Z}[G])$. From 8.2 we need to show that

$$\prod_{p \in S} |\varepsilon(\mathcal{Y}_p)|_p \varepsilon(\mathcal{Y}_p) \sim \prod_{p \in S} \varepsilon(b_p) |\varepsilon(b_p)|_p. \quad (37)$$

With the notation of 8.1 we know that for each prime $p \in S$,

$$\begin{aligned} |\varepsilon(\mathcal{Y}_p)|_p \varepsilon(\mathcal{Y}_p) &= |\varepsilon(\mathcal{Y}_p)|_p (\varepsilon(\mathcal{Y}_p)_p \times \prod_{q \neq p} \varepsilon(\mathcal{Y}_p)_q) \\ &= |\varepsilon(U_p)|_p \varepsilon(U_p)_p |\varepsilon(b_p)|_p \varepsilon(b_p)_p \times \prod_{q \neq p} \varepsilon(U_p)_q \varepsilon(b_p)_q. \end{aligned}$$

But from Theorem 8.1 we know $|\varepsilon(U_p)|_p \varepsilon(U_p)_p \sim 1$ and $\varepsilon(U_p)_q \sim 1$ whenever $q \neq p$. This then establishes (37), as required. \square

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